

---

Li W, Fearnhead P. [On the asymptotic efficiency of approximate Bayesian computation estimators](#). *Biometrika* (2018)

**DOI link**

<https://doi.org/10.1093/biomet/asx078>

**ePrints link**

[http://eprint.ncl.ac.uk/pub\\_details2.aspx?pub\\_id=241617](http://eprint.ncl.ac.uk/pub_details2.aspx?pub_id=241617)

**Date deposited**

29/01/2018

**Embargo release date**

20/01/2019

**Copyright**

This is a pre-copyedited, author-produced version of an article accepted for publication in *Biometrika* following peer review. The version of record is available online at:

<https://doi.org/10.1093/biomet/asx078>

# ON THE ASYMPTOTIC EFFICIENCY OF ABC ESTIMATORS

BY WENTAO LI AND PAUL FEARNHEAD

*Lancaster University*

Many statistical applications involve models for which it is difficult to evaluate the likelihood, but relatively easy to sample from. Approximate Bayesian computation (ABC) is a likelihood-free method for implementing Bayesian inference in such cases. It avoids evaluating the likelihood function and generates samples from an approximate posterior distribution by jointly simulating the parameter and the data, and accepting parameter values that give simulated data close to the observed data. We present results on the asymptotic variance of ABC estimators in a large-data limit. Our key assumption is that we summarise the data by a fixed dimensional summary statistic and that this summary statistic obeys a central limit theorem. We prove asymptotic normality of the ABC posterior mean. This improves on recent results on consistency for the ABC posterior mean, and in particular specifies its rate of convergence. This result also shows that, in terms of asymptotic variance, we should use a summary statistic that is the same dimension as the parameter vector,  $p$ ; and that any summary statistic of higher dimension can be reduced, through a linear transformation, to dimension  $p$  in a way that can only reduce the asymptotic variance of the ABC posterior mean. We then look at how the Monte Carlo error of an importance sampling algorithm that samples from the ABC posterior effects the accuracy of the ABC estimator. We give conditions on the importance sampling proposal distribution such that the variance of the ABC estimator will be the same order as that of the MLE based on the summary statistics used by ABC. This result suggests an iterative importance sampling algorithm, which we then evaluate empirically on a stochastic volatility model.

**1. Introduction.** There are many statistical applications which involve inference about models that are easy to simulate from, but for which it is difficult, or impossible, to calculate likelihoods. In such situations it is possible to use the fact we can simulate from the model to enable us to perform inference. There is a wide class of such *likelihood-free* methods of inference including indirect inference [19, 20], the bootstrap filter [18], simulated methods of moment [16], and synthetic likelihood [36].

We consider a Bayesian version of these methods, termed Approximate Bayesian Computation (ABC). This approach involves defining an approximation to the posterior distribution in such a way that it is possible to sample from this approximate posterior using

---

*Keywords and phrases:* Approximate Bayesian computation, Asymptotics, Importance Sampling, Partial Information, Proposal Distribution, Dimension Reduction

**Algorithm 1: Importance and Rejection Sampling ABC**

1. Simulate  $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N \sim q_n(\boldsymbol{\theta})$ ;
2. For each  $i = 1, \dots, N$ , simulate  $\mathbf{Y}^{(i)} = (y_1^{(i)}, \dots, y_n^{(i)}) \sim f_n(y|\boldsymbol{\theta}_i)$ ;
3. For each  $i = 1, \dots, N$ , accept  $\boldsymbol{\theta}_i$  with probability  $K_\varepsilon(\mathbf{s}_n^{(i)} - \mathbf{s}_{obs})$ , where  $\mathbf{s}_n^{(i)} = \mathbf{s}_n(\mathbf{Y}^{(i)})$ ; and define the associated weight as  $w_i = \pi(\boldsymbol{\theta}_i)/q_n(\boldsymbol{\theta}_i)$ .

only the ability to sample from the model. Arguably the first ABC method was that of [29], and these methods have been popular within population genetics [5, 11, 34], ecology [3] and systematic biology [33, 30]. More recently, there have been applications of ABC to other areas including stereology [9], stochastic differential equations [27], finance [26] and cosmology [22].

Let  $K(\mathbf{x})$  be a density kernel, where  $\max_{\mathbf{x}} K(\mathbf{x}) = 1$ , and  $\varepsilon > 0$  be a bandwidth. Denote the data as  $\mathbf{Y}_{obs} = (y_{obs,1}, \dots, y_{obs,n})$ . Assume we have chosen a finite dimensional summary statistic  $\mathbf{s}_n(\mathbf{Y})$ , and denote  $\mathbf{s}_{obs} = \mathbf{s}_n(\mathbf{Y}_{obs})$ . If we model the data as a draw from a parametric density,  $f_n(\mathbf{y}|\boldsymbol{\theta})$ , and assume prior,  $\pi(\boldsymbol{\theta})$ , then we define the ABC posterior as

$$(1) \quad \pi_{ABC}(\boldsymbol{\theta}|\mathbf{s}_{obs}, \varepsilon) \propto \pi(\boldsymbol{\theta}) \int f_n(\mathbf{s}_{obs} + \varepsilon \mathbf{v}|\boldsymbol{\theta}) K(\mathbf{v}) d\mathbf{v},$$

where  $f_n(\mathbf{s}|\boldsymbol{\theta})$  is the density for the summary statistic implied by  $f_n(\mathbf{y}|\boldsymbol{\theta})$ . Let  $f_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}, \varepsilon) = \int f_n(\mathbf{s}_{obs} + \varepsilon \mathbf{v}|\boldsymbol{\theta}) K(\mathbf{v}) d\mathbf{v}$ . This framework encompasses most implementations of ABC. In particular, the use of the uniform kernel corresponds to the popular rejection-based rule for ABC algorithm [5].

The idea is that  $f_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}, \varepsilon)$  is an approximation of the likelihood. The ABC posterior, which is proportional to the prior multiplied by this likelihood approximation, is an approximation of the true posterior. The likelihood approximation can be interpreted as a measure of, on average, how close the summary,  $\mathbf{s}_n$ , simulated from the model is to the summary for the observed data,  $\mathbf{s}_{obs}$ . The choices of kernel and bandwidth affect the definition of “closeness”.

By defining the approximate posterior in this way, we can simulate samples from it using standard Monte Carlo methods. One approach, that we will focus on later, uses importance sampling. Let  $K_\varepsilon(\mathbf{x}) = K(\mathbf{x}/\varepsilon)$ . Given a proposal density,  $q_n(\boldsymbol{\theta})$ , a bandwidth,  $\varepsilon$ , and a Monte Carlo sample size,  $N$ , the importance sampling ABC (IS-ABC) would proceed as in Algorithm 1. The set of accepted parameters and their associated weights provides a Monte Carlo approximation to  $\pi_{ABC}$ . Note that if we set  $q_n(\boldsymbol{\theta}) = \pi(\boldsymbol{\theta})$  then this is just a rejection sampler with the ABC posterior as its target, which is called rejection ABC in this paper. In practice sequential importance sampling methods are often used to learn a good proposal distribution [4].

There are three choices in implementing ABC: the choice of summary statistic, the choice of bandwidth, and the specifics of the Monte Carlo algorithm. For importance sampling,

the last of these involves specifying the Monte Carlo sample size,  $N$ , and the proposal density,  $q_n(\boldsymbol{\theta})$ . These, roughly, relate to three sources of approximation in ABC. To see this note that as  $\varepsilon \rightarrow 0$  we would expect ABC posterior to converge to the posterior given  $\mathbf{s}_{obs}$  [17]. Thus the choice of summary statistic governs the approximation, or loss of information, between using the full posterior distribution and using the posterior given the summary. The value  $\varepsilon$  then affects how close the ABC posterior is to the posterior given the summary. Finally there is then Monte Carlo error from approximating the true ABC posterior with a Monte Carlo sample. The Monte Carlo error is not only affected by the specifics of the Monte Carlo algorithm, but also by the choices of summary statistic and bandwidth, which together affect, say, the probability of acceptance in step 3 of the above importance sampling algorithm. Having a higher dimensional summary statistic, or a smaller value of  $\varepsilon$ , will tend to reduce this acceptance probability and hence increase the Monte Carlo error.

This work aims to study the interaction between all three sources of error, in the case where the summary statistics obey a central limit theorem (CLT) for large  $n$ . We are particularly interested in the efficiency of ABC, where by efficiency we mean that ABC has the same rate of convergence as the MLE for the parameter given the summary statistic. In particular this work is motivated by the question of whether ABC can be efficient as  $n \rightarrow \infty$  if we have a fixed Monte Carlo sample size. Intuitively this appears unlikely. For efficiency we will need  $\varepsilon \rightarrow 0$  as  $n \rightarrow \infty$ , and this corresponds to an increasingly strict condition for acceptance. Thus we may imagine that the acceptance probability will necessarily tend to 0 as  $n$  increases, and thus we will need an increasing Monte Carlo sample size to compensate for this.

However our results show that IS-ABC can be efficient if we choose an appropriate proposal distribution. The proposal distribution needs to have a suitable scale and location and have appropriately heavy tails. This can be achieved through an iterative procedure that learns the location and scale of the ABC posterior, and uses these as the basis of location and scale parameters for, say, a t-distributed proposal distribution. If we use an appropriate proposal distribution and have a summary statistic of the same dimension as the parameter vector we obtain that the ABC posterior mean is asymptotically unbiased with a variance that is  $1 + O(1/N)$  times that of the MLE based on the summary. This is similar to asymptotic results for indirect inference [19, 20], an alternative likelihood-free method. Our results also lend theoretical support to methods that choose the bandwidth indirectly through specifying the proportion of samples that are accepted. This approach leads to a bandwidth which is of the optimal order in  $n$ .

To obtain this result we first prove a Bernstein-von Mises type theorem for the ABC posterior mean. This is a non-standard convergence result as it is based on the partial information contained in the summary statistics. For related convergence results see [10] and [37]. However, this earlier work does not consider the case when the dimension of the summary statistic is larger than that of the parameter, which is commonplace in real-life applications of ABC. Dealing with a summary statistic of higher dimension than the

parameter vector introduces extra technical challenges. This is because previous proofs, based on the density of the summary be generalised as we now require densities on the lower dimensional manifold that is generated by the projecting from the summary to the parameter [37].

The convergence result we obtain for the ABC posterior mean has two practically important consequences. The first is that it shows that any  $d$  dimensional summary with  $d > p$  can be reduced to a  $p$  dimensional summary statistic without any loss of information: in that the posterior mean based on the reduced summary has the same asymptotic distribution as that based on the original summary. Furthermore it shows that using ABC with a summary statistic of dimension  $d > p$  can lead to an increased bias. This in turn means that the asymptotic variance of the ABC posterior mean can be larger than if the reduced summary was used. This advantage of using a summary of dimension  $p$  complements previous arguments for such a choice [17], which were based around reducing Monte Carlo variance.

This paper adds to a growing literature looking at the theoretical properties of ABC. Initial results focussed on the bias of ABC, defined as  $\mathbf{h}_{ABC} - \mathbb{E}[\mathbf{h}(\boldsymbol{\theta})|\mathbf{Y}_{obs}]$  where  $\mathbf{h}_{ABC}$  is the ABC posterior mean, and the Monte Carlo variance of estimating  $\mathbf{h}_{ABC}$ . The bias converges to  $\mathbb{E}[\mathbf{h}(\boldsymbol{\theta})|\mathbf{s}_{obs}] - \mathbb{E}[\mathbf{h}(\boldsymbol{\theta})|\mathbf{Y}_{obs}]$  as the bandwidth decreases to 0, hence the ABC estimator is consistent if  $\mathbf{s}_{obs}$  is sufficient [1, 17, 14]. The convergence rate of the bias is found to be as small as  $O(\varepsilon^2)$  in various settings [13, 1]. These results can then be used to consider how the choice of  $\varepsilon$  should depend on the Monte Carlo sample size so as to trade-off ABC bias and Monte Carlo variability [7, 1, 6]. They have also been used to give conditions for the ABC bias to be negligible when compared to the asymptotic variance of posterior mean, and to guide the selection of the bandwidth. For example, when the observations are i.i.d insights from [35] suggest that an upper bound for the bandwidth is  $o(n^{-1/2})$ ; whilst for observations from a hidden Markov model, [13] relaxed the upper bound to be  $o(n^{-1/4})$  in the case where the full dataset is used as the summary statistic. [13] further shows that the ABC posterior distribution can be arbitrarily close to the true posterior distribution as the bandwidth goes to 0 if the full-data is used as a summary statistic.

There has also been work looking at the consistency of ABC as we obtain more data. [23] consider consistency in ABC model choice and [24] consider consistency of ABC for parameter estimation. Both results shows consistency under weaker assumptions than we make. However our Theorem 3.1 gives a rate of convergence for ABC, shows how this depends on the choice of bandwidth, and shows how the asymptotic variance depends on the summary statistics. Also here it is natural to focus on the assumption similar to the classical Bernstein -von Mises theorem, since our purpose is to compare the efficiency with the likelihood-based estimators.

Finally, a number of papers have looked at the choice of summary statistics [e.g. 34, 25, 17, 2, 7, 28]. Whilst this is not the focus of our paper, Theorem 3.1 does give insight into this choice. As mentioned above, this result shows that, in terms of minimising the

asymptotic variance, we should use a summary that is of the same dimension as the number of parameters. In particular it further supports the suggestion in [17] of having one summary per parameter, with that summary approximating the MLE for that parameter (viewed as a function of the data). If we were to use the true MLEs as the summaries, then it follows from Theorem 3.1 that asymptotically the ABC posterior mean would attain the Cramer-Rao lower bound.

The paper is organised as follows. Section 2 sets up some notations and presents the key assumptions for the main theorems. Section 3 gives the asymptotic normality of the ABC posterior mean of  $\mathbf{h}(\boldsymbol{\theta})$  for  $n \rightarrow \infty$ . Section 4.1 gives the asymptotic normality of  $\hat{\mathbf{h}}$  when  $N \rightarrow \infty$ . In Section 4.2, the relative asymptotic efficiency between the MLE based on the summary statistic and  $\hat{\mathbf{h}}$  is studied for various proposal densities. An iterative importance sampling algorithm is proposed and the comparison between ABC and the indirect inference (II) is given. In Section 5 we demonstrate our results empirically on an analytically tractable normal example and a stochastic volatility model. paper concludes with some discussions. Proofs are contained in the Appendices.

**2. Notation and Set-up.** Denote the data by  $\mathbf{Y}_{obs} = (y_{obs,1}, \dots, y_{obs,n})$ , where  $n$  is the sample size, and each observation,  $y_{obs,i}$ , can be of arbitrary dimension. We will be considering the asymptotics as  $n \rightarrow \infty$ , and thus denote the density of  $\mathbf{Y}_{obs}$  by  $f_n(\mathbf{y}|\boldsymbol{\theta})$ . This density depends on an unknown parameter  $\boldsymbol{\theta}$ . We will let  $\boldsymbol{\theta}_0$  denote the true parameter value, and  $\pi(\boldsymbol{\theta})$  the prior distribution for the parameter. Let  $p$  be the dimension of  $\boldsymbol{\theta}$  and  $\mathcal{P}$  be the parameter space. For a set  $A$ , let  $A^c$  be its complement with respect to the whole space.

We assume that  $\boldsymbol{\theta}_0$  is in the interior of the parameter space:

(C1) There exists some  $\delta_0 > 0$ , such that  $\mathcal{P}_0 \equiv \{\boldsymbol{\theta} : |\boldsymbol{\theta} - \boldsymbol{\theta}_0| < \delta_0\} \subset \mathcal{P}$ .

To implement ABC we will use a summary statistic of the data,  $\mathbf{s}_n(\mathbf{Y}) \in \mathbb{R}^d$ ; for example a vector of sample means of appropriately chosen functions of the data. This summary statistic will be of fixed dimension,  $d$ , as we vary  $n$ . The density for  $\mathbf{s}_n(\mathbf{Y})$ , implied by the density for the data, will depend on  $n$ , and we denote this by  $f_n(\mathbf{s}|\boldsymbol{\theta})$ . We will use the shorthand  $\mathbf{S}_n$  to denote the random variable with density  $f_n(\mathbf{s}|\boldsymbol{\theta})$ . In ABC we use a kernel,  $K(\mathbf{x})$ , with  $\max_{\mathbf{x}} K(\mathbf{x}) = 1$ , and a bandwidth  $\varepsilon > 0$ . As we vary  $n$  we will often wish to vary  $\varepsilon$ , and in these situations denote the bandwidth by  $\varepsilon_n$ . For the importance sampling algorithm we require a proposal distribution,  $q_n(\boldsymbol{\theta})$ , and allow for this to depend on  $n$ . We assume the following conditions on the kernel:

- (C2) (i)  $\int \mathbf{v} K(\mathbf{v}) d\mathbf{v} = 0$ ;
- (ii)  $\int \prod_{k=1}^l v_{i_k} K(\mathbf{v}) d\mathbf{v} < \infty$  for any coordinates  $(v_{i_1}, \dots, v_{i_l})$  of  $\mathbf{v}$  and  $l \leq p + 6$ ;
- (iii)  $K(\mathbf{v}) = K(\|\mathbf{v}\|_{\Lambda}^2)$  where  $\|\mathbf{v}\|_{\Lambda}^2 = \mathbf{v}^T \Lambda \mathbf{v}$  and  $\Lambda$  is a diagonal matrix, and  $K(\mathbf{v})$  is a decreasing function of  $\|\mathbf{v}\|_{\Lambda}$ ;
- (iv)  $K(\mathbf{v}) = O(e^{-c_1 \|\mathbf{v}\|^{\alpha_1}})$  for some  $\alpha_1 > 0$  and  $c_1 > 0$  as  $\|\mathbf{v}\| \rightarrow \infty$ ,

which are satisfied by all commonly used kernels in ABC.

For a real function  $g(\mathbf{x})$  denote its  $k_{th}$  partial derivative at  $\mathbf{x} = \mathbf{x}_0$  by  $D_{x_k}g(\mathbf{x}_0)$ , the gradient function by  $D_{\mathbf{x}}g(\mathbf{x}_0)$  and the Hessian matrix by  $H_{\mathbf{x}}g(\mathbf{x}_0)$ . To simplify the notations,  $D_{\theta_k}$ ,  $D_{\boldsymbol{\theta}}$  and  $H_{\boldsymbol{\theta}}$  are written as  $D_k$ ,  $D$  and  $H$  respectively. For a series  $x_n$ , besides the limit notations  $O(\cdot)$  and  $o(\cdot)$ , we use the notations that for large enough  $n$ ,  $x_n = \Theta(a_n)$  if there exists constants  $m$  and  $M$  such that  $0 < m < |x_n/a_n| < M < \infty$ , and  $x_n = \Omega(a_n)$  if  $|x_n/a_n| \rightarrow \infty$ . For two square matrices  $A$  and  $B$ , we say  $A \leq B$  if  $B - A$  is semi-positive definite.

The asymptotic results are based around assuming a CLT for the summary statistic.

(C3) There exists a sequence  $a_n$ , with  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ , a  $d$ -dimensional vector  $\mathbf{s}(\boldsymbol{\theta})$  and a  $d \times d$  matrix  $A(\boldsymbol{\theta})$ , such that for all  $\boldsymbol{\theta} \in \mathcal{P}$ ,

$$a_n(\mathcal{S}_n - \mathbf{s}(\boldsymbol{\theta})) \xrightarrow{\mathcal{L}} N(0, A(\boldsymbol{\theta})); \text{ as } n \rightarrow \infty.$$

Furthermore, that

- (i)  $\mathbf{s}(\boldsymbol{\theta})$  and  $A(\boldsymbol{\theta}) \in C^1(\mathcal{P}_0)$ , and  $A(\boldsymbol{\theta})$  is positive definite for any  $\boldsymbol{\theta}$ ;
- (ii)  $\mathbf{s}(\boldsymbol{\theta}) = \mathbf{s}(\boldsymbol{\theta}_0)$  if and only if  $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ ; and
- (iii)  $I(\boldsymbol{\theta}) \triangleq D\mathbf{s}(\boldsymbol{\theta})^T A^{-1}(\boldsymbol{\theta}) D\mathbf{s}(\boldsymbol{\theta})$  has full rank at  $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ .

Under condition (C3) we have that  $a_n$  is the rate of convergence in the central limit theorem. If the data are independent and identically distributed, and the summaries consist of sample means of functions of the data, then  $a_n = n^{1/2}$ . Part (ii) of this condition is required for the true parameter to be identifiable given only the summary of data. Furthermore,  $I^{-1}(\boldsymbol{\theta}_0)/a_n^2$  is the asymptotic variance of the MLE based on the summary (henceforth MLES) for  $\boldsymbol{\theta}$  and therefore is required to be valid at the true parameter.

We next require a condition that controls the difference between  $f_n(\mathbf{s}|\boldsymbol{\theta})$  and its limiting distribution for  $\boldsymbol{\theta} \in \mathcal{P}_0$ . Let  $N(\mathbf{x}; \boldsymbol{\mu}, \Sigma)$  be the normal density at  $\mathbf{x}$  with mean  $\boldsymbol{\mu}$  and variance  $\Sigma$ . Define  $\tilde{f}_n(\mathbf{s}|\boldsymbol{\theta}) = N(\mathbf{s}; \mathbf{s}(\boldsymbol{\theta}), A(\boldsymbol{\theta})/a_n^2)$  and the standardization  $W_n(\mathbf{s}) = a_n A(\boldsymbol{\theta})^{-1/2}(\mathbf{s} - \mathbf{s}(\boldsymbol{\theta}))$ . Let  $f_{W_n}(\mathbf{w}|\boldsymbol{\theta})$  and  $\tilde{f}_{W_n}(\mathbf{w}|\boldsymbol{\theta})$  be the density of  $W_n(\mathbf{s})$  when  $\mathbf{s} \sim f_n(\mathbf{s}|\boldsymbol{\theta})$  and  $\tilde{f}_n(\mathbf{s}|\boldsymbol{\theta})$  respectively. The condition requires that the difference between  $f_{W_n}(\mathbf{w}|\boldsymbol{\theta})$  and its Edgeworth expansion  $\tilde{f}_{W_n}(\mathbf{w}|\boldsymbol{\theta})$  is  $o(a_n^{-2/5})$ , which is weaker than the standard requirement,  $o(a_n^{-1})$ , of the remainder from edgeworth expansion, and can be bounded by a density with exponentially decreasing tails. Specifically, assume that

(C4) there exists  $\alpha_n$  satisfying  $\alpha_n/a_n^{2/5} \rightarrow \infty$  and a density  $r_{max}(\mathbf{w})$  satisfying the same conditions as (C2) (ii)-(iv), such that  $\sup_{\boldsymbol{\theta} \in \mathcal{P}_0} \alpha_n |f_{W_n}(\mathbf{w}|\boldsymbol{\theta}) - \tilde{f}_{W_n}(\mathbf{w}|\boldsymbol{\theta})| \leq c_3 r_{max}(\mathbf{w})$  for some positive constant  $c_3$ .

For  $\boldsymbol{\theta}$  outside  $\mathcal{P}_0$ , the following condition requires the tails of  $f_n(\mathbf{s}|\boldsymbol{\theta})$  are also exponentially decreasing.

(C5)  $\sup_{\boldsymbol{\theta} \in \mathcal{P}_0^c} f_{W_n}(\mathbf{w}|\boldsymbol{\theta}) = O(e^{-c_2 \|\mathbf{w}\|^{\alpha_2}})$  as  $\|\mathbf{w}\| \rightarrow \infty$  for some positive constants  $c_2$  and  $\alpha_2$ , and  $A(\boldsymbol{\theta})$  is upper bounded in  $\mathcal{P}$ .



The differentiability of the prior density around the true parameter is required.

(C6)  $\pi(\boldsymbol{\theta}) \in C^1(\mathcal{P}_0)$  and  $\pi(\boldsymbol{\theta}_0) > 0$ .

Finally, the function of interest,  $\mathbf{h}(\boldsymbol{\theta})$ , needs to satisfy some differentiable and moment conditions in order that the remainders of its posterior moment expansion are small. Consider the  $k_{th}$  coordinate  $h_k(\boldsymbol{\theta})$  of  $\mathbf{h}(\boldsymbol{\theta})$ .

(C7)  $h_k(\boldsymbol{\theta}) \in C^1(\mathcal{P}_0)$  and  $D_k h(\boldsymbol{\theta}_0) \neq 0$ .

(C8)  $\int |h_k(\boldsymbol{\theta})| \pi(\boldsymbol{\theta}) d\boldsymbol{\theta} < \infty$  and  $\int h_k(\boldsymbol{\theta})^2 \pi(\boldsymbol{\theta}) d\boldsymbol{\theta} < \infty$ .

**3. Asymptotics of  $\mathbf{h}_{ABC}$ .** We first ignore the Monte Carlo error of ABC, and focus on the ideal ABC estimator,  $\mathbf{h}_{ABC}$ , where  $\mathbf{h}_{ABC} = E_{\pi_{ABC}}[\mathbf{h}(\boldsymbol{\theta})|\mathbf{s}_{obs}, \varepsilon_n]$ . As an approximation to the true posterior mean,  $E[\mathbf{h}(\boldsymbol{\theta})|\mathbf{Y}_{obs}]$ ,  $\mathbf{h}_{ABC}$  contains the errors from the choice of the bandwidth,  $\varepsilon_n$ , and the summary statistic  $\mathbf{s}_{obs}$ .

To understand the effect of these two sources of error, we derive results for the asymptotic distributions of  $\mathbf{h}_{ABC}$  and the likelihood-based estimators, including the MLE based on the summary (MLES) and the summary-based posterior mean, where we consider randomness solely due to the randomness of the data.

**THEOREM 3.1.** *Assume conditions (C1)–(C8).*

(i) Let  $\hat{\boldsymbol{\theta}}_{MLES} = \operatorname{argmax}_{\boldsymbol{\theta} \in \mathcal{P}} \log f_n(\mathbf{s}_{obs}|\boldsymbol{\theta})$  be the MLES of the parameter. For  $\mathbf{h}_{\mathbf{s}} = \mathbf{h}(\hat{\boldsymbol{\theta}}_{MLES})$  or  $E[\mathbf{h}(\boldsymbol{\theta})|\mathbf{s}_{obs}]$ , as  $n \rightarrow \infty$ , it holds that

$$a_n(\mathbf{h}_{\mathbf{s}} - \mathbf{h}(\boldsymbol{\theta}_0)) \xrightarrow{\mathcal{L}} N(0, D\mathbf{h}(\boldsymbol{\theta}_0)^T I^{-1}(\boldsymbol{\theta}_0) D\mathbf{h}(\boldsymbol{\theta}_0)).$$

(ii) If  $\varepsilon_n = o(a_n^{-3/5})$ , as  $n \rightarrow \infty$ , there exists a positive definite matrix  $I_{ABC}(\boldsymbol{\theta}_0)$  such that

$$a_n(\mathbf{h}_{ABC} - \mathbf{h}(\boldsymbol{\theta}_0)) \xrightarrow{\mathcal{L}} N(0, D\mathbf{h}(\boldsymbol{\theta}_0)^T I_{ABC}^{-1}(\boldsymbol{\theta}_0) D\mathbf{h}(\boldsymbol{\theta}_0)).$$

If  $\varepsilon_n = o(a_n^{-1})$  or  $d = p$  or  $A(\boldsymbol{\theta}_0)$  is diagonal,  $I_{ABC}(\boldsymbol{\theta}_0) = I(\boldsymbol{\theta}_0)$ . For other cases,  $I_{ABC}(\boldsymbol{\theta}_0) \leq I(\boldsymbol{\theta}_0)$ .

Theorem 3.1 (i) illustrates the validity of posterior inference based on the summary statistics. Regardless of the sufficiency and dimension of  $\mathbf{s}_{obs}$ , the posterior mean based on the summary statistics is consistent and asymptotically normal with the same variance as that of the MLES. This is similar to the equivalence of the posterior mean and MLE based on the full dataset implied by the classical Bernstein-von Mises theorem.

Theorem 3.1 (ii) indicates three types of bandwidth choice which determine three cases for the approximation accuracy of the ABC posterior mean. Denote the ABC bias,  $\mathbf{h}_{ABC} - E[\mathbf{h}(\boldsymbol{\theta})|\mathbf{s}_{obs}]$ , by  $\text{bias}_{ABC}$ . The first case is ‘negligible’  $\varepsilon_n$ , which is when  $\varepsilon_n$  is  $o(1/a_n)$  and for which  $\text{bias}_{ABC}$  is negligible. This conforms to the result implied by [35] that if  $\varepsilon_n = o(1/a_n)$ , the wrong model likelihood adopted by ABC is the same as the true likelihood to the first



order. The second case is ‘dominating’  $\varepsilon_n$  which is when  $\varepsilon_n$  is  $\Theta(1/\sqrt{a_n})$  or  $\Omega(1/\sqrt{a_n})$ . Although not formally stated here, it is expected that  $\text{bias}_{ABC}$  is dominating in this case, making the convergence rate of  $\mathbf{h}_{ABC}$  slower than  $a_n$ . The third case is ‘well-behaved’  $\varepsilon_n$  which is between the previous two cases and for which the convergence rate is  $a_n$ . Furthermore,  $\text{bias}_{ABC}$  is still negligible if the dimension of the summary statistics is equal to that of the parameter. However if the dimension is larger  $\mathbf{h}_{ABC}$  can be less efficient than MLES. the ‘negligible’  $\varepsilon_n$  is preferred, as we see below the Monte Carlo acceptance rate will inevitably degenerate as  $n \rightarrow \infty$ , and the required Monte Carlo size would need to increase with  $n$ .

$d > p$ , Theorem 3.1 (ii) shows that  $\text{bias}_{ABC}$  is non-negligible and increases the asymptotic variance. This is essentially because the leading term of  $\text{bias}_{ABC}$  is proportional to the average of  $\mathbf{v} = \mathbf{s} - \mathbf{s}_{obs}$ , the difference between the simulated and observed summary statistics, and if  $d > p$ , the marginal density of  $\mathbf{v}$  is generally asymmetric, and thus has a non-zero mean.

It has previously been argued that one should choose a summary statistic which has the same dimension as the number of parameters [17]. However that was based on controlling the Monte Carlo error, with for example [8] showing that the optimal rate of decreasing  $\epsilon$  as the Monte Carlo sampling size increases is slower for larger  $d$ . The loss of efficiency we observe in Theorem 3.1 (ii) for  $d > p$  gives a separate advantage for choosing a summary statistic with  $d = p$ . Remarkably, the following proposition shows that for any summary statistic of dimension  $d > p$  we can find a new  $p$ -dimensional summary statistic without any loss of information.

**PROPOSITION 3.1.** *Assume the conditions of Theorem 3.1. If  $d$  is larger than  $p$ , let  $C = D\mathbf{s}(\boldsymbol{\theta}_0)^T A(\boldsymbol{\theta}_0)^{-1}$ , then  $I_C(\boldsymbol{\theta}_0) = I(\boldsymbol{\theta}_0)$  where  $I_C(\boldsymbol{\theta})$  is the  $I(\boldsymbol{\theta})$  matrix of the summary statistic  $C\mathbf{S}_n$ . Therefore the asymptotic variance of  $\mathbf{h}_{ABC}$  based on  $C\mathbf{s}_{obs}$  is smaller than or equal to that based on  $\mathbf{s}_{obs}$ .*

**PROOF.** The equality can be verified by algebra. □

The proposition shows that a proper linear transformation can be an effective dimension reduction method, when  $\varepsilon_n$  is small enough that the condition in Theorem 3.1 (ii) is satisfied. The matrix  $C$  can be interpreted as the product of the scale matrix  $A(\boldsymbol{\theta}_0)^{-1/2}$ , which standardizes  $\mathbf{s}_{obs}$ , and the matrix  $D\mathbf{s}(\boldsymbol{\theta}_0)^T A(\boldsymbol{\theta}_0)^{-1/2}$  which can be taken as the ‘squared-root’ of  $I(\boldsymbol{\theta}_0)$ .

Theorem 3.1 leads to following natural definition.

**DEFINITION 1.** *Assume that the conditions of Theorem 3.1 hold. Then the asymptotic variance of  $\mathbf{h}_{ABC}$  is*

$$AV_{\mathbf{h}_{ABC}} = \frac{1}{a_n^2} D\mathbf{h}(\boldsymbol{\theta}_0)^T I_{ABC}^{-1}(\boldsymbol{\theta}_0) D\mathbf{h}(\boldsymbol{\theta}_0).$$

#### 4. Asymptotic Properties of Rejection and Importance Sampling ABC.

4.1. *Asymptotic Monte Carlo Error.* We now consider the Monte Carlo error involved in estimating  $\mathbf{h}_{ABC}$ . Here we fix the data and consider randomness solely in terms of the stochasticity of the Monte Carlo algorithm. We focus on the importance sampling algorithm given in the introduction. Remember that  $N$  is the Monte Carlo sample size. For  $i = 1, \dots, N$ ,  $\boldsymbol{\theta}_i$  is the proposed parameter value and  $w_i$  is its importance sampling weight. Let  $\phi_i$  be the indicator that is 1 if and only if  $\boldsymbol{\theta}_i$  is accepted in step 3 of algorithm 1 and  $N_{acc} = \sum_{i=1}^N \phi_i$  be the number of accepted parameter.

Provided  $N_{acc} \geq 1$  we can estimate  $\mathbf{h}_{ABC}$  from the output of importance sampling algorithm with

$$\hat{\mathbf{h}} = \sum_{i=1}^N \mathbf{h}(\boldsymbol{\theta}_i) w_i \phi_i / \sum_{i=1}^N w_i \phi_i.$$

Define

$$p_{acc,q} = \int q(\boldsymbol{\theta}) \int f_n(\mathbf{s}|\boldsymbol{\theta}) K_\varepsilon(\mathbf{s} - \mathbf{s}_{obs}) d\mathbf{s} d\boldsymbol{\theta},$$

which is the acceptance probability of the importance sampling algorithm proposing from  $q(\boldsymbol{\theta})$ . Furthermore, define

$$q_{ABC}(\boldsymbol{\theta}|\mathbf{s}_{obs}, \varepsilon) \propto q_n(\boldsymbol{\theta}) f_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}, \varepsilon),$$

the density of the accepted parameter; and

$$\begin{aligned} \Sigma_{IS,n} &\equiv E_{\pi_{ABC}} \left[ (\mathbf{h}(\boldsymbol{\theta}) - \mathbf{h}_{ABC})^2 \frac{\pi_{ABC}(\boldsymbol{\theta}|\mathbf{s}_{obs}, \varepsilon_n)}{q_{ABC}(\boldsymbol{\theta}|\mathbf{s}_{obs}, \varepsilon_n)} \right] \\ (2) \quad \text{and } \Sigma_{ABC,n} &\equiv p_{acc,q_n}^{-1} \Sigma_{IS,n}, \end{aligned}$$

where  $\Sigma_{IS,n}$  is the IS variance with  $\pi_{ABC}$  as the target density and  $q_{ABC}$  as the proposal density. Note that  $p_{acc,q_n}$  and  $\Sigma_{IS,n}$ , and hence  $\Sigma_{ABC,n}$ , depend on  $\mathbf{s}_{obs}$ .

Standard results give the following asymptotic distribution of  $\hat{\mathbf{h}}$ .

PROPOSITION 4.1. *For a given  $n$  and  $\mathbf{s}_{obs}$ , if  $\mathbf{h}_{ABC}$  and  $\Sigma_{ABC,n}$  are finite, then*

$$\sqrt{N}(\hat{\mathbf{h}} - \mathbf{h}_{ABC}) \xrightarrow{\mathcal{L}} N(0, \Sigma_{ABC,n}),$$

as  $N \rightarrow \infty$ .

The proposition motivates the following definition.

DEFINITION 2. *For a given  $n$  and  $\mathbf{s}_{obs}$ , assume that the conditions of Proposition 4.1 hold. Then the asymptotic Monte Carlo variance of  $\hat{\mathbf{h}}$  is*

$$MCV_{\hat{\mathbf{h}}} = \frac{1}{N} \Sigma_{ABC,n}.$$

From Proposition 4.1, it can be seen that the asymptotic Monte Carlo variance of  $\hat{\mathbf{h}}$  is equal to the IS variance  $\Sigma_{IS,n}$  divided by the average number of acceptance  $Np_{acc,q_n}$ , and therefore depends on the proposal distribution and  $\varepsilon_n$  through these two terms.

**4.2. Asymptotic efficiency.** We have defined the asymptotic variance as  $n \rightarrow \infty$  of  $\mathbf{h}_{ABC}$ , and the asymptotic Monte Carlo variance, as  $N \rightarrow \infty$  of  $\hat{\mathbf{h}}$ . Both the error of  $\mathbf{h}_{ABC}$  when estimating  $\mathbf{h}(\boldsymbol{\theta}_0)$  and the Monte Carlo error of  $\hat{\mathbf{h}}$  when estimating  $\mathbf{h}_{ABC}$  are independent of each other. Thus this suggests the following definition.

**DEFINITION 3.** Assume the conditions of Theorem 3.1, and that  $\mathbf{h}_{ABC}$  and  $\Sigma_{ABC,n}$  are bounded in probability for any  $n$ . Then the asymptotic variance of  $\hat{\mathbf{h}}$  is

$$AV_{\hat{\mathbf{h}}} = \frac{1}{a_n^2} \mathbf{h}(\boldsymbol{\theta}_0)^T I_{ABC}^{-1}(\boldsymbol{\theta}_0) D\mathbf{h}(\boldsymbol{\theta}_0) + \frac{1}{N} \Sigma_{ABC,n}.$$

That is the asymptotic variance of  $\hat{\mathbf{h}}$  is the sum of its Monte Carlo asymptotic variance for estimating  $\mathbf{h}_{ABC}$ , and the asymptotic variance of  $\mathbf{h}_{ABC}$ .

We now wish to investigate the properties of this asymptotic variance, for large but fixed  $N$ , as  $n \rightarrow \infty$ . In particular we are interested in the ratio between  $AV_{\hat{\mathbf{h}}}$  and  $AV_{MLES}$ , where, by Theorem 3.1, the latter is defined as  $a_n^{-2} \mathbf{h}(\boldsymbol{\theta}_0)^T I^{-1}(\boldsymbol{\theta}_0) D\mathbf{h}(\boldsymbol{\theta}_0)$ . We will consider how this ratio depends on the choice of  $\varepsilon_n$  and  $q_n(\boldsymbol{\theta})$ . Thus we introduce the following definition:

**DEFINITION 4.** For a choice of  $\varepsilon_n$  and  $q_n(\boldsymbol{\theta})$ , we define the asymptotic efficiency of  $\hat{\mathbf{h}}$  as

$$AE_{\hat{\mathbf{h}}} = \lim_{n \rightarrow \infty} \frac{AV_{MLES}}{AV_{\hat{\mathbf{h}}}}.$$

If this limiting value is 0, we say that  $\hat{\mathbf{h}}$  is asymptotically inefficient.

We will investigate the asymptotic efficiency of  $\hat{\mathbf{h}}$  under the assumption of Theorem 3.1 that  $\varepsilon_n = o(1/\sqrt{a_n})$ . We will see that the convergence rate of the IS variance  $\Sigma_{IS,n}$  depends on how large  $\varepsilon_n$  is, and so we further define  $c_\varepsilon = \lim_{n \rightarrow \infty} a_n \varepsilon_n$ , assuming that this limit exists, and let  $a_{n,\varepsilon} = a_n \mathbb{1}_{c_\varepsilon < \infty} + \varepsilon_n^{-1} \mathbb{1}_{c_\varepsilon = \infty}$ . Note that  $c_\varepsilon$  can be either a constant or infinity.

First we show that if we propose from the prior or the posterior, then the ABC estimator is asymptotically inefficient.

**THEOREM 4.1.** Assume the conditions of Theorem 3.1. we have:

- (i) If  $q_n(\boldsymbol{\theta}) = \pi(\boldsymbol{\theta})$ ,  $p_{acc,q_n} = \Theta_p(\varepsilon_n^d a_{n,\varepsilon}^{d-p})$  and  $\Sigma_{IS,n} = \Theta_p(a_{n,\varepsilon}^{-2})$ .
- (ii) If  $q_n(\boldsymbol{\theta}) = \pi_{ABC}(\boldsymbol{\theta} | \mathbf{s}_{obs}, \varepsilon_n)$ ,  $p_{acc,q_n} = \Theta_p(\varepsilon_n^d a_{n,\varepsilon}^d)$  and  $\Sigma_{IS,n} = \Theta_p(a_{n,\varepsilon}^p)$ .

In both cases  $\hat{\mathbf{h}}$  are asymptotically inefficient.

Note the result in part (ii) shows the difference from standard importance sampling settings, where using the target distribution as the proposal leads to an estimator with no Monte Carlo error.

The reason why  $\hat{\mathbf{h}}$  is asymptotically inefficient is because the Monte Carlo variance decays more slowly than  $1/a_n^2$  as  $n \rightarrow \infty$ . However the problem with the Monte Carlo variance is caused by different factors in each case.

To see this, consider the acceptance probability of a value of  $\theta$  and corresponding summary  $\mathbf{s}_n$  simulated in one iteration of the IS-ABC algorithm. This acceptance probability depends on

$$(3) \quad \frac{\mathbf{s}_n - \mathbf{s}_{obs}}{\varepsilon_n} = \frac{1}{\varepsilon_n} [(\mathbf{s}_n - \mathbf{s}(\theta)) + (\mathbf{s}(\theta) - \mathbf{s}(\theta_0)) + (\mathbf{s}(\theta_0) - \mathbf{s}_{obs})],$$

where  $\mathbf{s}(\theta)$ , defined in (C3), is the limiting values of  $\mathbf{s}_n$  as  $n \rightarrow \infty$  if data is sampled from the model for parameter value  $\theta$ . By (C3) the first and third bracketed terms within the square brackets on the right-hand side are  $O_p(a_n^{-1})$ . If we sample from the prior, then the middle term is  $O_p(1)$ , and thus (3) will blow-up as  $\varepsilon_n$  goes to 0. Hence  $p_{acc,\pi}$  goes to 0 as  $\varepsilon_n$  goes to 0 and thus causes the estimate to be inefficient. If we sample from the posterior, then by Theorem 3.1 we expect the middle term to also be  $O_p(a_n^{-1})$ . Hence (3) is well behaved as  $n \rightarrow \infty$ , and consequently  $p_{acc,\pi}$  is bounded away from 0, provided either  $\varepsilon_n = \Theta(a_n^{-1})$  or  $\varepsilon_n = \Omega(a_n^{-1})$ .

However, using  $\pi_{ABC}(\theta|\mathbf{s}_{obs}, \varepsilon_n)$  as a proposal distribution still causes the estimate to be inefficient due to an increasing variance of the importance weights. As  $n$  increases the proposal is more and more concentrated around  $\theta_0$ , while  $\pi$  does not change. Therefore the weight, which is the ratio of  $\pi_{ABC}$  and  $q_{ABC}$ , is increasingly skewed and causes  $\Sigma_{IS,n}$  to go to  $\infty$ .

discussed after Theorem 3.1, when  $\varepsilon_n = o(a_n^{-1})$ , its effect on the bias is negligible. However, for any Monte Carlo algorithm making acceptance/rejection through  $K(\mathbf{v})$ , the acceptance probability with this choice of  $\varepsilon_n$  goes to 0 as  $n \rightarrow \infty$ . Because in (3), the mechanism simulating the dataset determines that  $\mathbf{s}_n - \mathbf{s}_{obs}$  is  $O_p(a_n^{-1})$  and hence with the negligible  $\varepsilon_n$ , (3) will blow-up, making the acceptance probability degenerate. In such a case,  $N$  needs to increase with  $n$  to compensate the decreasing acceptance rate.

**4.3. Efficient Proposal Distributions.** Whilst using the prior and the posterior leads to asymptotically inefficient estimators, it will be seen that there exist practical proposal distributions that avoid this inefficiency. Consider proposing the parameter value from a location-scale family. That is our proposal is of the form  $\sigma_n \Sigma^{1/2} \mathbf{X} + \mu_n$ , where  $\mathbf{X} \sim q(\cdot)$ ,  $E[\mathbf{X}] = 0$  and  $Var[\mathbf{X}] = I_p$ . This defines a general form of proposal density, where the center,  $\mu_n$ , the scale rate,  $\sigma_n$ , the scale matrix,  $\Sigma$  and the base density,  $q(\cdot)$ , all need to be specified. We will give conditions under which such a proposal density results in estimators that are not inefficient.

Our results are based on an expansion of  $\pi_{ABC}(\theta|\mathbf{s}_{obs}, \varepsilon_n)$ , obtained from the proof of Lemma 6 in the Appendix. Consider the rescaled random variables  $\mathbf{t} = a_{n,\varepsilon}(\theta - \theta_0)$  and

$\mathbf{v} = \varepsilon_n^{-1}(\mathbf{s} - \mathbf{s}_{obs})$ . Let  $\mathbf{T}_{obs} = a_n A(\boldsymbol{\theta}_0)^{-1/2}(\mathbf{s}_{obs} - \mathbf{s}(\boldsymbol{\theta}_0))$ . Define a joint density of  $\mathbf{t}$  and  $\mathbf{v}$  as the following,

$$g_n(\mathbf{t}, \mathbf{v}; \tau) \propto \begin{cases} N\left((D\mathbf{s}(\boldsymbol{\theta}_0) + \tau)\mathbf{t}; a_n \varepsilon_n \mathbf{v} + A(\boldsymbol{\theta}_0)^{1/2} \mathbf{T}_{obs}, A(\boldsymbol{\theta}_0)\right) K(\mathbf{v}), & \text{when } a_n \varepsilon_n \rightarrow c < \infty, \\ N\left((D\mathbf{s}(\boldsymbol{\theta}_0) + \tau)\mathbf{t}; \mathbf{v} + \frac{1}{a_n \varepsilon_n} A(\boldsymbol{\theta}_0)^{1/2} \mathbf{T}_{obs}, \frac{1}{a_n^2 \varepsilon_n^2} A(\boldsymbol{\theta}_0)\right) K(\mathbf{v}), & \text{when } a_n \varepsilon_n \rightarrow \infty, \end{cases}$$

and  $g_n(\mathbf{t}; \tau) \triangleq \int g_n(\mathbf{t}, \mathbf{v}; \tau) d\mathbf{v}$ . These are defined so that for large  $n$  and for the rescaled variables, the leading term of  $\pi_{ABC}$  is proportional to  $g_n(\mathbf{t}; 0)$ . Note that this is a continuous mixture of normal density with the kernel density as the weights.

The main theorem here requires two conditions of the proposal density. First, we need that the density for the scaled random variables,  $\mathbf{t}$ , to be proper, which requires  $\sigma_n = a_{n,\varepsilon}^{-1}$  and is  $O_p(1)$ . This ensures that the acceptance probability is bounded away from 0. Second we need the importance ratio between the target and the proposal densities satisfies the following:

(C10)  $\exists \alpha \in (0, 1)$  and a small enough  $\delta > 0$  such that, for any  $\boldsymbol{\mu}$  bounded in probability,

$$\sup_{\mathbf{t} \in \mathbb{R}^p, \boldsymbol{\tau}^T \boldsymbol{\tau} \leq \delta I_p} \frac{g_n(\mathbf{t}; \tau)^\alpha}{q(\Sigma^{-1/2}(\mathbf{t} - \boldsymbol{\mu}))} = O_p(1).$$

if we further choose  $\varepsilon_n = \Theta(a_n^{-1})$ , the Monte Carlo IS variance for the accepted parameter values is  $\Theta(a_n^{-2})$ , and has the same order as the variance of MLES.

THEOREM 4.2. *Assume the conditions of Theorem 3.1. If*

$$q_n(\boldsymbol{\theta}) = \beta \pi(\boldsymbol{\theta}) + (1 - \beta) \frac{1}{\sigma_n^p |\Sigma|^{1/2}} q(\sigma_n^{-1} \Sigma^{-1/2}(\boldsymbol{\theta} - \boldsymbol{\mu}_n)),$$

where  $\beta \in (0, 1)$ ,  $q(\cdot)$  and  $\Sigma$  satisfy (C10),  $\sigma_n = a_{n,\varepsilon}^{-1}$  and it holds that  $p_{acc,q_n} = \Theta_p(\varepsilon_n^d a_{n,\varepsilon}^d)$  and  $\Sigma_{IS,n} = O_p(a_{n,\varepsilon}^{-2})$ . Then if  $\varepsilon_n = \Theta(a_n^{-1})$ ,  $AE_{\hat{\mathbf{h}}} = \Theta_p(1)$ .

Furthermore, if  $d = p$ ,  $AE_{\hat{\mathbf{h}}} = 1 - K/(N + K)$  for some constant  $K$ .

The mixture with  $\pi(\boldsymbol{\theta})$  here is to control the importance weight in the tail area, similarly to the defensive importance sampling of [21]. It is not clear whether this is needed in practice, or is just a consequence of the approach taken in the proof.

The above result also shows that with a good proposal distribution, if the acceptance probability is bounded away from 0 as  $n$  increases, the threshold  $\varepsilon_n$  will have the preferred rate  $\Theta(a_n^{-1})$ . This supports the intuitive idea of using the acceptance rate in ABC to choose the threshold based on aiming for an appropriate proportion of acceptances [e.g. 15, 6].

In practice, obviously  $\sigma_n$  and  $\mu_n$  need to be adaptive to the observations since they depend on  $n$ . For  $q(\cdot)$  and  $\Sigma$ , the following proposition gives a practical suggestion satisfying (C10). Let  $T(\cdot; \gamma)$  be the multivariate  $t$  density with degree of freedom  $\gamma$ .

**Algorithm 2: Iterative Importance Sampling ABC**

Choose a small mixture weight  $\beta$ , and a sequence of acceptance rates  $\{p_k\}$ . Choose a location-scale family (such as a  $t$ -distribution). Let  $q_0$  be the density from this family that has the same mean and variance as the prior.

At the  $k_{\text{th}}$  step:

1. Run IS-ABC with simulation size  $N_0$ , proposal density  $\beta\pi(\boldsymbol{\theta}) + (1 - \beta)q_k(\boldsymbol{\theta})$  and acceptance rate  $p_k$ , and record the bandwidth  $\varepsilon_k$ .
2. If  $\varepsilon_{k-1} - \varepsilon_k$  is smaller than some positive threshold, stop. Otherwise, let  $\mu_{k+1}$  and  $\Sigma_{k+1}$  be the empirical mean and variance matrix of the weighted sample from step 1, and let  $q_{k+1}(\boldsymbol{\theta})$  be the density with centre  $\mu_{k+1}$  and variance matrix  $2\Sigma_{k+1}$ .
3. If  $q_k(\boldsymbol{\theta})$  is close to  $q_{k+1}(\boldsymbol{\theta})$ , stop. Otherwise, return to step 1.

After the iteration stops at the  $K_{\text{th}}$  step, run the IS-ABC with the proposal density  $\beta\pi(\boldsymbol{\theta}) + (1 - \beta)q_{K+1}(\boldsymbol{\theta})$ ,  $N - KN_0$  simulations and  $p_{K+1}$ .

**PROPOSITION 4.2.** *If  $\exists \gamma_0 > 0$  and  $\alpha \in (0, 1)$  such that  $K(\mathbf{v})^\alpha / T(\mathbf{v}; \gamma_0) \leq M$  for some constant  $M$ , then (C10) is satisfied for  $q(\boldsymbol{\theta}) = T(\boldsymbol{\theta}; \gamma)$ , where  $\gamma \leq \gamma_0$  and any  $\Sigma$ .*

The above result says that it is theoretically valid to choose any  $\Sigma$  if a  $t$  distribution with any choice of  $\gamma$  is chosen as the base density, providing the kernel has lighter tails than the  $t$ -distribution.

**4.4. Iterative Importance Sampling ABC.** Taken together, Theorem 4.2 and Proposition 4.2 suggest proposing from the mixture of  $\pi(\boldsymbol{\theta})$  and a  $t$  distribution with the scale matrix and center approximating those of  $\pi_{ABC}(\boldsymbol{\theta})$ . We suggest using an iterative procedure [similar in spirit to that of 4], see Algorithm 2.

In this algorithm,  $N$  is the number of simulations allowed by the computing budget,  $N_0 < N$  and  $\{p_k\}$  is a sequence of acceptance rate, which we use to choose the bandwidth. The rule for choosing the new proposal distribution is based on approximating the mean and variance of the density proportional to  $\pi(\boldsymbol{\theta})f_{ABC}(\mathbf{s}_{\text{obs}}|\boldsymbol{\theta}, \varepsilon)^{1/2}$ , which is optimal in the sense of maximising the ESS of importance sampling [17]. It can be shown that these two moments are approximately equal to the mean and twice the variance of  $\pi_{ABC}(\boldsymbol{\theta})$  respectively. the mixture weight,  $\beta$ , we suggest using 0.05. Since Algorithm 2 has the same simulation size as the rejection ABC and the additional calculations have negligible computational cost, the iterative procedure does not introduce additional computational

## 5. Numerical Examples.

**5.1. Gaussian Likelihood with Sample Quantiles.** This examples illustrates the results in Section 3 with an analytically tractable problem. Assume the observations  $\mathbf{Y}_{\text{obs}} = (y_1, \dots, y_n)$  follow the univariate normal distribution  $N(\mu, \sigma)$  with true parameter values  $(1, \sqrt{2})$ . Consider estimating the unknown parameter  $(\mu, \sigma)$  with the uniform prior in the area  $[-10, 10] \times [-10, 10]$  using Algorithm 1. The summary statistic implemented in

algorithm 1 is  $(e^{\hat{q}_{\alpha_1}/2}, \dots, e^{\hat{q}_{\alpha_d}/2})$  where  $\hat{q}_{\alpha}$  is the sample quantile of  $\mathbf{Y}_{obs}$  for probability  $\alpha$ . Since the likelihood function and asymptotic distribution of the summary statistic are analytically available [31], the theoretical results in Theorem 3.1 and Proposition 3.1 may be verified. This summary statistic is illuminating because it is easy to change the information contained by changing the number of quantiles and it avoids the trivial case that  $s(\theta)$  is a linear function of  $\theta$ .

The results for data size  $n = 10^5$  are presented. Smaller sizes from  $10^2$  to  $10^4$  also have been tested, and all show similar patterns. The probabilities  $\alpha_1, \dots, \alpha_d$  for calculating quantiles are selected with equal intervals in  $(0, 1)$ , and  $d = 2, 4, 9$  and 19 are tested. In order to investigate the Monte Carlo error-free performance,  $N$  is chosen to be large enough. The performance of  $\theta_{ABC}$ , MLES and MLE are compared. Since the dimension reduction matrix  $C$  in Proposition 3.1 can be obtained analytically, the performance of  $\theta_{ABC}$  using the original  $d$ -dimension summary is compared with that using the 2-dimension summary. results of mean square error(MSE) are presented in Figure 1.

The phenomena implied by Theorem 3.1 and Proposition 3.1 can be seen in this example, together with the limitations of these results. First,  $E[\mathbf{h}(\theta)|\mathbf{s}_{obs}]$ , equivalent to  $\theta_{ABC}$  with small enough  $\varepsilon$ , and *MLES* have the same performance. Second, the three stages of increasing  $\varepsilon$  can be seen in all graphs. When  $\varepsilon$  is small, the MSEs of  $\theta_{ABC}$  achieve those of *MLES*. Then when  $\varepsilon$  becomes larger, for  $d > 2$  the MSEs quickly increase to be significantly larger than those of the *MLES*, while for  $d = 2$  there are no such obvious gaps. This corresponds to the ‘well-behaved’  $\varepsilon$ . Then the increasing rates of MSEs become larger as the  $\varepsilon$  increases and becoming more and more ‘dominating’.

Third, for all cases, the 2-dimension summary give the same performance of  $\theta_{ABC}$  as the *MLES* for small  $\varepsilon$ , indicating that it contains the same information as the original summary. However, it be seen that for larger  $\varepsilon$ , the performance of the reduced-dimension summaries are not stable, and are in fact worse than the original summaries for estimating  $\mu$ , although better for estimating  $\sigma$ . The worse/better performance are caused larger/smaller bias of  $\theta_{ABC}$ . This is due to the second order behaviour of  $\theta_{ABC}$ , which becomes important for larger  $\varepsilon$ . This suggests using other techniques for reducing the bias, e.g. the regression adjustment, together with the dimension-reduction matrix for more stable behaviour.

**5.2. Stochastic Volatility with AR(1) Dynamics.** Consider the stochastic volatility model in [32]

$$\begin{cases} x_n &= \phi x_{n-1} + \eta_n, \quad \eta_n \sim N(0, \sigma_\eta^2) \\ y_n &= \bar{\sigma} e^{\frac{x_n}{2}} \xi_n, \quad \xi_n \sim N(0, 1), \end{cases}$$

where  $\eta_n$  and  $\xi_n$  are independent,  $y_n$  is the demeaned return of a portfolio obtained by subtracting the average of all returns from the actual return and  $\bar{\sigma}$  is the average volatility level. By the transformation  $y_n^* = \log y_n^2$  and  $\xi_n^* = \log \xi_n^2$ , the state-space model can be



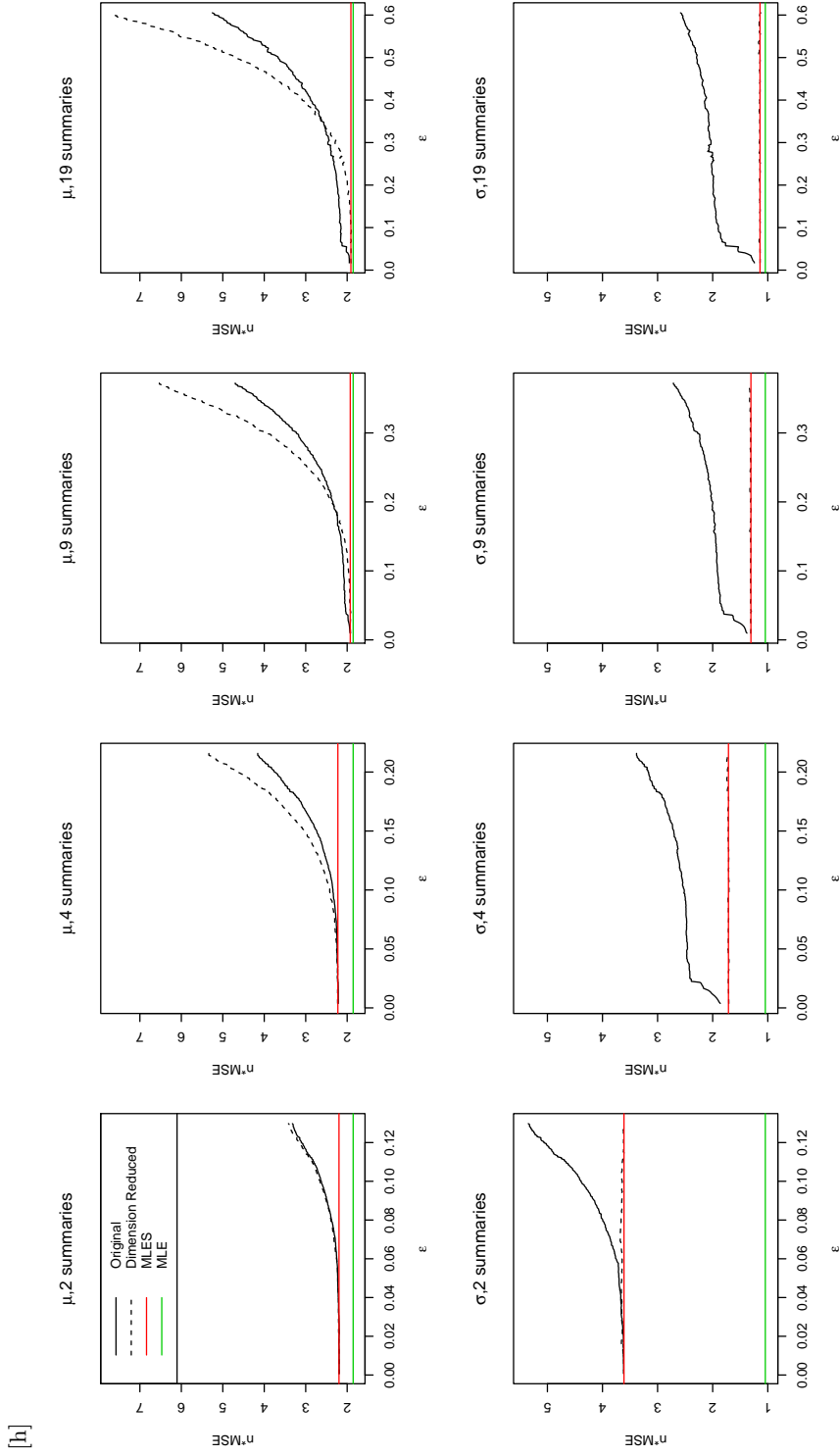


FIG 1. Illustration of the results in Section 3. MSE multiplying by  $n$  are reported and the experiment is replicated for 200 times. ‘Original’ stands for  $\theta_{ABC}$  based on the original summary statistics and for ‘Dimension reduced’, the summary statistics are linearly transformed to 2-dimension according to Proposition 3.1.

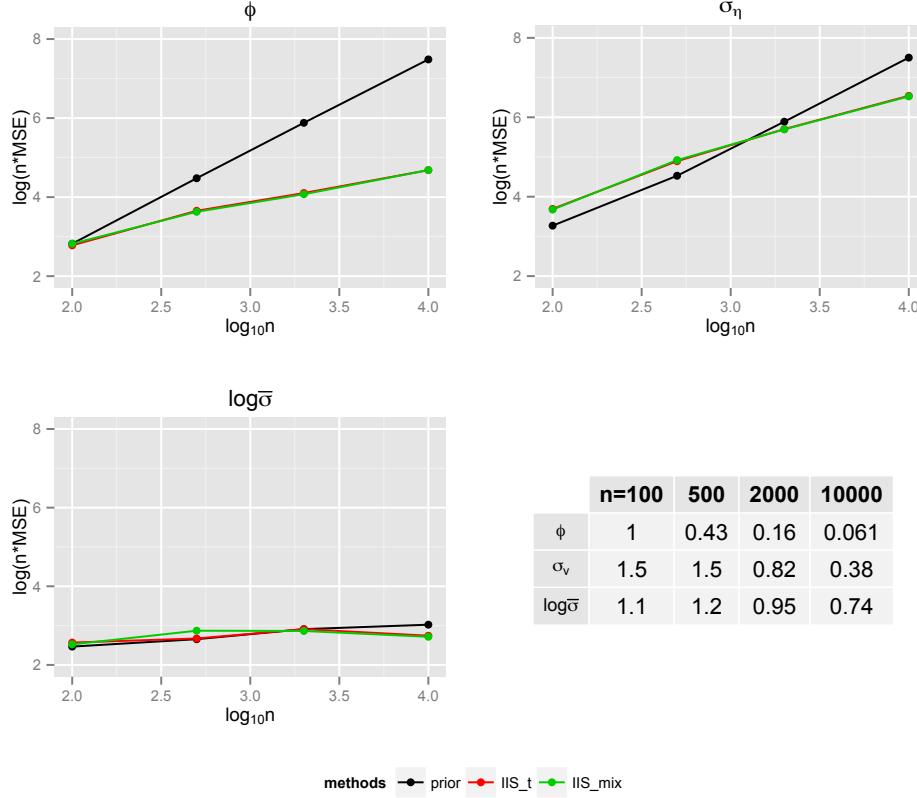


FIG 2. Comparisons of R-ABC and IIS-ABC with two implementation for increasing  $n$ . For IIS-mix, the estimation stage uses the mixture as the proposal distribution, and for IIS-t,  $t$  distribution only. For each  $n$ , the logarithm of average MSE for 100 datasets multiplying by  $n$  is reported. For each dataset, the Monte Carlo sample size of ABC estimators is  $10^4$ . The ratio of the MSEs of the two methods is given in the table, and smaller values indicate better performance of the IIS-ABC.

transformed to

$$(4) \quad \begin{cases} x_n &= \phi x_{n-1} + \eta_n, \quad \eta_n \sim N(0, \sigma_\eta^2) \\ y_n^* &= 2 \log \bar{\sigma} + x_n + \xi_n^*, \quad \exp\{\xi_n^*\} \sim \chi_1^2, \end{cases}$$

which is linear and non-Gaussian.

The ABC method can be used to obtain an off-line estimator for the unknown parameter of the state-space models, which is recently discussed by [24]. Here we illustrate the effectiveness of iteratively choosing the importance proposal for large  $n$  by comparing the performance of the rejection ABC (R-ABC) and the iterative IS-ABC. In the iterative algorithm,  $t$  distribution with degree of freedom 5 is used to construct  $q_k$ . In order to see whether it is necessary to bound the skewed importance weights using mixture, we implement the final estimation using two proposal distributions, the mixture  $\beta\pi(\boldsymbol{\theta}) + (1 - \beta)q_{K+1}(\boldsymbol{\theta})$  and  $q_{K+1}(\boldsymbol{\theta})$  only.

Consider the estimation of the parameter  $(\phi, \sigma_\eta, \log \bar{\sigma})$  with the uniform prior in the area  $[0, 1] \times [0.1, 3] \times [-10, -1]$ . The setting with the true parameter  $(\phi, \sigma_\eta, \log \bar{\sigma}) = (0.9, 0.675, -4.1)$  is studied, which is motivated by the empirical studies. For any dataset  $\mathbf{Y} = (y_1, \dots, y_n)$ , let  $\mathbf{Y}^* = (y_1^*, \dots, y_n^*)$ . The summary statistic  $\mathbf{s}_n(\mathbf{Y}) = (\widehat{Var}[\mathbf{Y}^*], \widehat{Cor}[\mathbf{Y}^*], \widetilde{E}[\mathbf{Y}^*])$  is used, where  $\widehat{Var}$ ,  $\widehat{Cor}$  and  $\widetilde{E}$  denote the empirical variance, lag-1 autocorrelation and mean. If there were no noise in the state equation for  $\xi_n^*$  in (4), then  $\mathbf{s}_n(\mathbf{Y})$  would be a sufficient statistic of  $\mathbf{Y}^*$ , and hence is a natural choice for the summary statistic. The uniform kernel is used in the accept-reject step of ABC.

The data length  $n = 100, 500, 2000$  and  $10000$  are tested with the simulation budget  $N = 10000$ . For the IIS-ABC, the sequence  $\{p_k\}$  has the first five values being 5% to 1%, decreasing by 1%, and the other values being 1%. For R-ABC, both 5% and 1% quantiles are tried and 5% is chosen for its better performance. For each iteration,  $N_0 = 1000$ . The simulation results are shown in Figure 2.

It can be seen that for all parameters, the IIS-ABC shows increasing advantage over the R-ABC as  $n$  increases. For larger  $n$ , the iterative procedure obtains the center of proposals closer to the true parameter and the bandwidth smaller than those used in the R-ABC, and the comparison becomes more significant when  $n$  increases. These contribute to the more accurate ABC estimators. For smaller  $n$ , both perform similarly, since when the summary statistic is not accurate enough, the ABC posterior is not much different from the prior, and the benefit of sampling from a slightly better proposal does not compensate the increased Monte Carlo variance from the importance weight. It is relatively easier to estimate  $\log \bar{\sigma}$ , since the summary statistic  $\widetilde{E}[\mathbf{Y}^*]$  is centered at a linear function of  $\log \bar{\sigma}$ , and therefore IIS-ABC does not show as much advantage over R-IIS as estimating  $\phi$  and  $\sigma_v$ . Finally, the performance both with and without the mixture for the proposal density are similar.

**6. Summary and Discussion.** The results in this paper suggest that ABC can scale to large data, at least for models with a fixed number of parameters. Under the assumption that the summary statistics obey a central limit theorem (as defined in Condition C3), then we have that asymptotically the ABC posterior mean of a function of the parameters is normally distributed about the true value of that function. The asymptotic variance of the estimator is equal to the asymptotic variance of the MLE for the function give the summary statistic. And without loss of asymptotic efficiency we can always use a summary statistic that has the same dimension as the number of parameters. This is a stronger result than that of [17], where they show that choosing the same number of summaries as parameters is optimal when interest is in estimating just the parameters.

We have further shown that appropriate importance sampling implementations of ABC are efficient, in the sense of increasing the asymptotic variance of our estimator by a factor that is just  $O(1/N)$ . However similar results are likely to apply to SMC and MCMC implementations of ABC. For example ABC-MCMC will be efficient provided the acceptance probability does not degenerate to 0 as  $n$  increases. However at stationarity, ABC-MCMC will propose parameter values from a distribution close to the ABC posterior density, and

Theorems 5.1 and 5.2 suggest that for such a proposal distribution the acceptance probability of ABC will be bounded away from 0.

Whilst our theoretical results suggest that point estimates based on the ABC posterior have good properties, they do not suggest that the ABC posterior is a good approximation to the true posterior, nor that the ABC posterior will accurately quantify the uncertainty in estimates. It can be shown from a simple Gaussian example that the ABC posterior will tend to over-estimate the uncertainty.

**Acknowledgements** This work was supported by the Engineering and Physical Sciences Research Council, grant EP/K014463.

## References.

- [1] Barber, S., J. Voss, and M. Webster (2013). The rate of convergence for approximate Bayesian computation. *arXiv preprint arXiv:1311.2038*.
- [2] Barnes, C. P., S. Filippi, M. P. Stumpf, and T. Thorne (2012). Considerate approaches to constructing summary statistics for abc model selection. *Statistics and Computing* 22(6), 1181–1197.
- [3] Beaumont, M. A. (2010). Approximate Bayesian computation in evolution and ecology. *Annual review of ecology, evolution, and systematics* 41, 379–406.
- [4] Beaumont, M. A., J.-M. Cornuet, J.-M. Marin, and C. P. Robert (2009). Adaptive approximate Bayesian computation. *Biometrika* 96(4), 983–990.
- [5] Beaumont, M. A., W. Zhang, and D. J. Balding (2002). Approximate Bayesian computation in population genetics. *Genetics* 162, 2025–2035.
- [6] Biau, G., F. Cérou, and A. Guyader (2012). New insights into approximate Bayesian computation. *arXiv preprint arXiv:1207.6461*.
- [7] Blum, M. G. (2010). Approximate Bayesian computation: a nonparametric perspective. *Journal of the American Statistical Association* 105(491).
- [8] Blum, M. G., M. A. Nunes, D. Prangle, S. A. Sisson, et al. (2013). A comparative review of dimension reduction methods in approximate bayesian computation. *Statistical Science* 28(2), 189–208.
- [9] Bortot, P., S. G. Coles, and S. A. Sisson (2007). Inference for stereological extremes. *Journal of the American Statistical Association* 102(477), 84–92.
- [10] Clarke, B. and J. Ghosh (1995). Posterior convergence given the mean. *Annals of statistics* 23(6), 2116–2144.
- [11] Cornuet, J.-M., F. Santos, M. A. Beaumont, C. P. Robert, J.-M. Marin, D. J. Balding, T. Guillemaud, and A. Estoup (2008). Inferring population history with DIY ABC: a user-friendly approach to approximate Bayesian computation. *Bioinformatics* 24(23), 2713–2719.
- [12] Creel, M. and D. Kristensen (2013). Indirect likelihood inference (revised). UFAE and IAE working papers, Unitat de Fonaments de l’Anàlisi Econòmica (UAB) and Institut d’Anàlisi Econòmica (CSIC).
- [13] Dean, T. A. and S. S. Singh (2011). Asymptotic behaviour of approximate Bayesian estimators. *arXiv preprint arXiv:1105.3655*.
- [14] Dean, T. A., S. S. Singh, A. Jasra, and G. W. Peters (2014). Parameter estimation for hidden Markov models with intractable likelihoods. *Scandinavian Journal of Statistics*.
- [15] Del Moral, P., A. Doucet, and A. Jasra (2012). An adaptive sequential Monte Carlo method for approximate Bayesian computation. *Statistics and Computing* 22(5), 1009–1020.
- [16] Duffie, D. and K. J. Singleton (1993). Simulated moments estimation of Markov models of asset prices. *Econometrica* 61(4), 929–952.
- [17] Fearnhead, P. and D. Prangle (2012). Constructing summary statistics for approximate Bayesian computation: semi-automatic approximate Bayesian computation. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 74(3), 419–474.

- [18] Gordon, N., D. Salmond, and A. F. M. Smith (1993). Novel approach to nonlinear/non-Gaussian Bayesian state estimation. *IEEE proceedings-F* 140, 107–113.
- [19] Gouriéroux, C. and E. Ronchetti (1993). Indirect inference. *Journal of Applied Econometrics* 8, s85–s118.
- [20] Heggland, K. and A. Frigessi (2004). Estimating functions in indirect inference. *Journal of the Royal Statistical Society, series B* 66, 447–462.
- [21] Hesterberg, T. (1995). Weighted average importance sampling and defensive mixture distributions. *Technometrics* 37(2), 185–194.
- [22] Ishida, E., S. Vitenti, M. Penna-Lima, J. Cisewski, R. de Souza, A. Trindade, E. Cameron, and V. Busti (2015/11//). cosmoabc: Likelihood-free inference via population monte carlo approximate bayesian computation. *Astronomy and Computing* 13.
- [23] Marin, J.-M., N. S. Pillai, C. P. Robert, and J. Rousseau (2013). Relevant statistics for Bayesian model choice. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*.
- [24] Martin, G. M., B. P. McCabe, W. Maneesoonthorn, and C. P. Robert (2014). Approximate Bayesian computation in state space models. *arXiv preprint arXiv:1409.8363*.
- [25] Nunes, M. A. and D. J. Balding (2010). On optimal selection of summary statistics for approximate bayesian computation. *Statistical applications in genetics and molecular biology* 9(1).
- [26] Peters, G. W., B. Kannan, B. Lasscock, C. Mellen, S. Godsill, et al. (2011). Bayesian cointegrated vector autoregression models incorporating alpha-stable noise for inter-day price movements via approximate Bayesian computation. *Bayesian Analysis* 6(4), 755–792.
- [27] Picchini, U. (2013). Inference for SDE models via approximate Bayesian computation. *Journal of Computational and Graphical Statistics* (just-accepted).
- [28] Prangle, D., P. Fearnhead, M. P. Cox, P. J. Biggs, and N. P. French (2014). Semi-automatic selection of summary statistics for abc model choice. *Statistical applications in genetics and molecular biology* 13(1), 67–82.
- [29] Pritchard, J. K., M. T. Seielstad, A. Perez-Lezaun, and M. W. Feldman (1999). Population growth of human Y chromosomes: a study of Y chromosome microsatellites. *Molecular Biology and Evolution* 16, 1791–1798.
- [30] Ratmann, O., C. Andrieu, C. Wiuf, and S. Richardson (2009). Model criticism based on likelihood-free inference, with an application to protein network evolution. *Proceedings of the National Academy of Sciences* 106(26), 10576–10581.
- [31] Reiss, R.-D. (2012). *Approximate distributions of order statistics: with applications to nonparametric statistics*. Springer Science & Business Media.
- [32] Sandmann, G. and S. Koopman (1998). Estimation of stochastic volatility models via Monte Carlo maximum likelihood. *Journal of Econometrics* 87(2), 271–301.
- [33] Toni, T., D. Welch, N. Strelkowa, A. Ipsen, and M. P. Stumpf (2009). Approximate Bayesian computation scheme for parameter inference and model selection in dynamical systems. *Journal of the Royal Society Interface* 6(31), 187–202.
- [34] Wegmann, D., C. Leuenberger, and L. Excoffier (2009). Efficient approximate Bayesian computation coupled with Markov chain Monte Carlo without likelihood. *Genetics*.
- [35] Wilkinson, R. D. (2013). Approximate Bayesian computation (ABC) gives exact results under the assumption of model error. *Statistical applications in genetics and molecular biology* 12(2), 129–141.
- [36] Wood, S. N. (2010). Statistical inference for noisy nonlinear ecological dynamic systems. *Nature* 466(7310), 1102–1104.
- [37] Yuan, A. and B. Clarke (2004). Asymptotic normality of the posterior given a statistic. *Canadian Journal of Statistics* 32(2), 119–137.

**Appendix.** Technical lemmas and proofs of the main results are presented. Throughout the appendix the data are considered to be random, and  $O(\cdot)$  and  $\Theta(\cdot)$  denote the lim-

iting behaviour when  $n$  goes to  $\infty$ . For two sets  $A$  and  $B$ , the sum of integrals  $\int_A f(\mathbf{x}) d\mathbf{x} + \int_B f(\mathbf{x}) d\mathbf{x}$  is written as  $(\int_A + \int_B)f(\mathbf{x}) d\mathbf{x}$ . Recall that  $\mathbf{T}_{obs} = a_n A(\boldsymbol{\theta}_0)^{-1/2}(\mathbf{s}_{obs} - \mathbf{s}(\boldsymbol{\theta}_0))$  and by (C3),  $\mathbf{T}_{obs} \xrightarrow{\mathcal{L}} N(0, I_d)$  where  $I_d$  is the identity matrix with dimension  $d$ . For a vector  $\mathbf{v}$ , denote any polynomial of the elements of  $\mathbf{v}$  with order up to  $l$  by  $P_l(\mathbf{v})$ . We say a square matrix  $A$  is bounded if there exists constants  $c_1$  and  $c_2$  such that  $c_1 \leq A \leq c_2$  and a rectangular matrix  $B$  is bounded if  $B^T B$  is bounded. Let  $\lambda_{min}(A) = c_1$  and  $\lambda_{max}(A) = c_2$ .

## APPENDIX A: PROOF OF SECTION 3

The proof of Theorem 3.1 proceeds as follows. The convergence of the MLES is given by Lemma 1 and Lemma 2. For the convergence of the posterior and ABC posterior means, divide  $\mathbb{R}^p$  into  $B_\delta = \{\boldsymbol{\theta} : \|\boldsymbol{\theta} - \boldsymbol{\theta}_0\| < \delta\}$  and  $B_\delta^c$  for some  $\delta < \delta_0$ . First, in  $B_\delta^c$ , Lemma 3 shows that the integration is ignorable. In  $B_\delta$ , the integral can be approximated by replacing  $f_n(\mathbf{s}_{obs}|\boldsymbol{\theta})$  with a normal density, suggested by (7). Then the expansions of the mean and the normalising constant of ABC posterior density, based on the analytical form of the normal density, are given in Lemma 6, and the vanishing of the remainder terms are supported by Lemma 4. Finally, the asymptotic distributions of the leading terms of the expansion are obtained according to Lemma 5, which concludes the proof.

For MLES, [12] gives the central limit theorem for  $\hat{\boldsymbol{\theta}}_{MLES}$  when  $a_n = \sqrt{n}$  and  $\mathcal{P}$  is compact. According to the proof in [12], extending the result to the general  $a_n$  is straightforward. Additionally, we give the extension for general  $\mathcal{P}$ .

LEMMA 1. Assume conditions (C1), (C3)-(C5). Then it holds that  $a_n(\hat{\boldsymbol{\theta}}_{MLES} - \boldsymbol{\theta}_0) \xrightarrow{\mathcal{L}} N(0, I^{-1}(\boldsymbol{\theta}_0))$  as  $n \rightarrow \infty$ .

Given the condition (C7), by Lemma 1 and the delta method, the convergence of MLES for general  $h(\boldsymbol{\theta})$  holds as the following.

LEMMA 2. Assume the conditions of Lemma 1 and (C7). Then

$$a_n(h(\hat{\boldsymbol{\theta}}_{MLES}) - h(\boldsymbol{\theta}_0)) \xrightarrow{\mathcal{L}} N(0, Dh(\boldsymbol{\theta}_0)^T I^{-1}(\boldsymbol{\theta}_0) Dh(\boldsymbol{\theta}_0)) \text{ as } n \rightarrow \infty.$$

Now consider the integral  $\pi(h) = \int h(\boldsymbol{\theta})\pi(\boldsymbol{\theta})f_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}) d\boldsymbol{\theta}$ . Under this notation,  $h_{ABC} = \pi(h)/\pi(1)$ . For some  $\delta < \delta_0$ , decompose  $\pi(h)$  into two parts, including

$$\pi_{B_\delta}(h) \triangleq \int_{B_\delta} h(\boldsymbol{\theta})\pi(\boldsymbol{\theta})f_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}) d\boldsymbol{\theta} \text{ and } \pi_{B_\delta^c}(h) \triangleq \int_{B_\delta^c} h(\boldsymbol{\theta})\pi(\boldsymbol{\theta})f_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}) d\boldsymbol{\theta}.$$

First of all, the following lemma shows that for a fixed  $\delta$ , the integral in  $B_\delta^c$  can be ignored.

LEMMA 3. Assume conditions (C2)(iii), (C4), (C5) and (C8). Then  $\forall \delta > 0$ ,  $\pi_{B_\delta^c}(h) = O_p(e^{-a_n^{\alpha_\delta} c_\delta})$  for some positive constants  $c_\delta$  and  $\alpha_\delta$  depending on  $\delta$ .

PROOF. It is sufficient to show that  $\sup_{\boldsymbol{\theta} \in B_\delta^c} f_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}) = O_p(e^{-a_{n,\varepsilon}^{\alpha_\delta} c_\delta})$ . Let  $M_\delta = \min(M_1, \delta)$ . By dividing  $\mathbb{R}^d$  into  $\{\mathbf{v} : \|\varepsilon_n \mathbf{v}\| \leq M_\delta/2\}$  and its complement, we have

$$\begin{aligned} & \sup_{\boldsymbol{\theta} \in B_\delta^c} \int_{\mathbb{R}^d} f_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v}|\boldsymbol{\theta}) K(\mathbf{v}) d\mathbf{v} \\ & \leq \sup_{\boldsymbol{\theta} \in B_\delta^c \setminus \mathcal{P}_0^c} \sup_{\|\mathbf{s} - \mathbf{s}_{obs}\| \leq M_\delta/2} f_n(\mathbf{s}|\boldsymbol{\theta}) + \sup_{\boldsymbol{\theta} \in \mathcal{P}_0^c} \sup_{\|\mathbf{s} - \mathbf{s}_{obs}\| \leq M_\delta/2} f_n(\mathbf{s}|\boldsymbol{\theta}) + K(\varepsilon_n^{-1} M_\delta/2) \varepsilon_n^{-d}. \end{aligned}$$

In the above, as  $n \rightarrow \infty$ , both the second and the third terms are exponentially decreasing by (C5) and (C2)(iii) respectively. For  $\boldsymbol{\theta} \in B_\delta^c \setminus \mathcal{P}_0^c$ , when  $\|\mathbf{s} - \mathbf{s}_{obs}\| \leq M_\delta/2$ ,  $\|W_n(\mathbf{s})\| \geq a_n \delta r$  for some constant  $r$ . Since  $f_{W_n}(\mathbf{w}|\boldsymbol{\theta})$  is bounded by the sum of a normal density and  $r_{max}(\mathbf{w})$ ,  $\sup_{\boldsymbol{\theta} \in B_\delta^c \setminus \mathcal{P}_0^c} \sup_{\|\mathbf{s} - \mathbf{s}_{obs}\| \leq M_\delta/2} f_n(\mathbf{s}|\boldsymbol{\theta})$  is also exponentially decreasing. Finally, the sum of all the above is  $O(-a_{n,\varepsilon}^{\alpha_\delta} c_\delta)$  by noting that  $a_{n,\varepsilon} \leq \min(\varepsilon_n^{-1}, a_n)$ .  $\square$

Then we only need to consider the integration in  $B_\delta$ . Let  $\mathbf{t}(\boldsymbol{\theta})$  be the rescaled random vector  $a_{n,\varepsilon}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)$  and  $t(B_\delta)$  be the transformed  $B_\delta$  under  $\mathbf{t}(\boldsymbol{\theta})$ . This rescaling is useful in the following Taylor expansion,

$$(5) \quad \frac{\pi_{B_\delta}(h)}{\pi_{B_\delta}(1)} = h(\boldsymbol{\theta}_0) + a_{n,\varepsilon}^{-1} D h(\boldsymbol{\theta}_0)^T \frac{\pi_{B_\delta}(t(\boldsymbol{\theta}))}{\pi_{B_\delta}(1)} + \frac{1}{2} a_{n,\varepsilon}^{-2} \frac{\pi_{B_\delta}(t(\boldsymbol{\theta}))^T H h(\boldsymbol{\theta}_0) t(\boldsymbol{\theta}))}{\pi_{B_\delta}(1)},$$

where  $\boldsymbol{\theta}_t \in B_\delta$ . Let  $r_n(\mathbf{s}|\boldsymbol{\theta})$  be the scaled remainder  $\alpha_n[f_n(\mathbf{s}|\boldsymbol{\theta}) - \tilde{f}_n(\mathbf{s}|\boldsymbol{\theta})]$ . Let  $\tilde{f}_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}) = \int \tilde{f}_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v}|\boldsymbol{\theta}) K(\mathbf{v}) d\mathbf{v}$  and  $\tilde{\pi}_{B_\delta}(h) \triangleq \int_{B_\delta} h(\boldsymbol{\theta}) \pi(\boldsymbol{\theta}) \tilde{f}_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}) d\boldsymbol{\theta}$ . Intuitively  $\pi_{B_\delta}(h)$  can be approximated by  $\tilde{\pi}_{B_\delta}(h)$  if their difference is small, written as the following

$$\pi_{B_\delta}(P_l(\mathbf{t})) - \tilde{\pi}_{B_\delta}(P_l(\mathbf{t})) = \alpha_n^{-1} \int_{t(B_\delta)} \int P_l(\mathbf{t}) \pi(\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t}) r_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v}|\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t}) K(\mathbf{v}) d\mathbf{v} d\mathbf{t},$$

For now we claim that

$$(6) \quad \frac{\int_{t(B_\delta)} \int P_l(\mathbf{t}, \mathbf{v}) \pi(\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t}) r_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v}|\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t}) K(\mathbf{v}) d\mathbf{v} d\mathbf{t}}{\int_{B_\delta} \pi(\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t}) \tilde{f}_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t}) d\mathbf{t}} = O_p(1),$$

and leave the proof to Lemma 7. Then it implies the expansion

$$(7) \quad \pi_{B_\delta}(1) = \tilde{\pi}_{B_\delta}(1)(1 + O_p(\alpha_n^{-1})) \text{ and } \frac{\pi_{B_\delta}(P_l(\mathbf{t}))}{\pi_{B_\delta}(1)} = \frac{\tilde{\pi}_{B_\delta}(P_l(\mathbf{t}))}{\tilde{\pi}_{B_\delta}(1)} + O_p(\alpha_n^{-1}).$$

The following two lemmas are given to analyse the convolutions involved in  $\tilde{\pi}_{B_\delta}(P_l(\mathbf{t}))$ .

LEMMA 4. Assume condition (C2). For  $\mathbf{t} \in \mathbb{R}^p$ , let  $A(\mathbf{t})$  be a  $d \times p$  matrix function. Let  $\mathbf{c}$  be a constant vector,  $\{k_n\}$  be a series converging to  $c_1 \in (0, \infty]$ , and  $\{b_n\}$  be a series converging to a non-negative constant. Let  $B_n = \mathbb{1}_{\{c_1 = \infty\}} + b_n \mathbb{1}_{\{c_1 < \infty\}}$ . Assume  $A(\mathbf{t})$  is uniformly bounded in  $\mathbb{R}^p$ . For a density  $g(\mathbf{v})$  in  $\mathbb{R}^d$ , if it satisfies



- (i)  $g(\mathbf{v}) = g(\|\mathbf{v}\|)$  and  $g(\mathbf{v})$  is a decreasing function of  $\|\mathbf{v}\|$ , and  
(ii)  $\int \prod_{k=1}^{l+p} v_{i_k} g(\mathbf{v}) d\mathbf{v} < \infty$  for any coordinates  $(v_{i_1}, \dots, v_{i_l})$  of  $\mathbf{v}$  for some integer  $l$ ,

then

$$\int_{\mathbb{R}^p} \int_{\mathbb{R}^d} P_l(\mathbf{t}, \mathbf{v}) k_n^d g(k_n[A(\mathbf{t})\mathbf{t} - B_n\mathbf{v} - k_n^{-1}\mathbf{c}]) K(\mathbf{v}) d\mathbf{v} d\mathbf{t} = O(1),$$

$$\text{and } \int_{\mathbb{R}^p} \int_{\mathbb{R}^d} k_n^d g(k_n[A(\mathbf{t})\mathbf{t} - B_n\mathbf{v} - k_n^{-1}\mathbf{c}]) K(\mathbf{v}) d\mathbf{v} d\mathbf{t} = \Theta(1).$$

PROOF. Note that  $K(\mathbf{v})$  satisfies (i) and (ii). When  $c_1 < \infty$ , assume  $k_n = 1$  without loss of generality. Divide  $\mathbb{R}^p$  into  $V = \{\mathbf{t} : \|A(\mathbf{t})\mathbf{t}\|/2 \geq \|b_n\mathbf{v} + \mathbf{c}\|\}$  and  $V^c$ . In  $V$ ,  $\|A(\mathbf{t})\mathbf{t} - b_n\mathbf{v} - \mathbf{c}\| \geq \|A(\mathbf{t})\mathbf{t}\|/2$ ; in  $V^c$ ,  $\|\mathbf{t}\| \leq 2\lambda_{\min}(A)^{-1}\|b_n\mathbf{v} + \mathbf{c}\|$ . Note that  $P_l(\mathbf{t}, \mathbf{v}) \leq P_l(\|\mathbf{t}\|, \|\mathbf{v}\|)$  by Cauchy-Schwarz inequality. Then

$$\begin{aligned} & \int_{\mathbb{R}^p} \int_{\mathbb{R}^d} P_l(\mathbf{t}, \mathbf{v}) g(A(\mathbf{t})\mathbf{t} - b_n\mathbf{v} - \mathbf{c}) K(\mathbf{v}) d\mathbf{v} d\mathbf{t} \\ & \leq \int_{\mathbb{R}^p} \int_{\mathbb{R}^d} P_l(\mathbf{t}, \mathbf{v}) g(\lambda_{\max}(A)\|\mathbf{t}\|/2) K(\mathbf{v}) d\mathbf{v} d\mathbf{t} \\ & \quad + c_2 \sup_{\mathbf{v} \in \mathbb{R}^d} g(\mathbf{v}) \int_{\mathbb{R}^p} \int_{\mathbb{R}^d} P_l(2\lambda_{\min}(A)^{-1}\|b_n\mathbf{v} + \mathbf{c}\|, \mathbf{v}) \|b_n\mathbf{v} + \mathbf{c}\|^p K(\mathbf{v}) d\mathbf{v} d\mathbf{t}, \\ & = O(1), \end{aligned}$$

where  $c_2$  is some constant.

When  $c_1 = \infty$ , let  $\mathbf{v}^* = k_n(A(\mathbf{t})\mathbf{t} - \mathbf{v} - k_n^{-1}\mathbf{c})$ . Then

$$\begin{aligned} & \int_{\mathbb{R}^p} \int_{\mathbb{R}^d} P_l(\mathbf{t}, \mathbf{v}) k_n^d g(k_n[A(\mathbf{t})\mathbf{t} - \mathbf{v} - k_n^{-1}\mathbf{c}]) K(\mathbf{v}) d\mathbf{v} d\mathbf{t} \\ & = \int_{\mathbb{R}^p} \int_{\mathbb{R}^d} P_l(\mathbf{t}, \mathbf{v}^*) K(A(\mathbf{t})\mathbf{t} - k_n^{-1}\mathbf{c} - k_n^{-1}\mathbf{v}^*) g(\mathbf{v}^*) d\mathbf{v}^* d\mathbf{t}, \end{aligned}$$

which is  $O(1)$  following the previous arguments.

For  $P_l(\mathbf{t}, \mathbf{v}) = 1$ , by considering only the integral in a compact region, it is easy to see the target integral is larger than 0. Therefore the lemma holds.  $\square$

LEMMA 5. Consider the notations and assumption of Lemma 4. Then

$$\int t \frac{N(A\mathbf{t}; B_n\mathbf{v} + \frac{1}{k_n}\mathbf{c}, \frac{1}{k_n^2}I_d) K(\mathbf{v})}{\int N(A\mathbf{t}; B_n\mathbf{v} + \frac{1}{k_n}\mathbf{c}, \frac{1}{k_n^2}I_d) K(\mathbf{v}) d\mathbf{t} d\mathbf{v}} d\mathbf{t} d\mathbf{v} = \frac{1}{k_n} \left[ (A^T A)^{-1} A^T \mathbf{c} + r(\mathbf{c}; A, B_n, k_n) \mathbb{1}_{\{d > p\}} \right].$$

It holds that (i)  $r(\mathbf{c}; A, B_n, k_n) = 0$  if  $d = p$ , and  $O(1)$  if  $d > p$ , and  $o(1)$  if  $B_n = o(1)$ . (ii)  $r(0; A, B, k_n) = 0$ .

The explicit expression of  $r(\mathbf{c}; A, B_n, k_n)$  is tedious and stated in the supplementary material.

Let  $g_n(\mathbf{t}, \mathbf{v}) = g_n(\mathbf{t}, \mathbf{v}; 0)$ , where  $g_n(\mathbf{t}, \mathbf{v}; \tau)$  is defined in Section 4.3, and for a  $d \times d$  matrix  $\tau_2$ , let  $g_n(\mathbf{t}, \mathbf{v}; \tau_1, \tau_2)$  be defined by replacing  $A(\boldsymbol{\theta}_0)$  in  $g_n(\mathbf{t}, \mathbf{v}; \tau_1)$  by  $(A(\boldsymbol{\theta}_0)^{-1/2} + \tau_2)^{-2}$ . The following lemma gives expansions for the leading term of the normalising constant  $\pi(1)$  and the posterior mean  $\pi(h)/\pi(1)$ , by showing that the leading term of  $\pi(\boldsymbol{\theta})\tilde{f}_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta})$  is proportional to  $g_n(\mathbf{t}, \mathbf{v})$ .

LEMMA 6. Assume conditions (C2)(ii), (C3), (C6) and (C7). If  $\varepsilon_n = o(1/\sqrt{a_n})$ , then it holds that

(8)

$$\pi_{B_\delta}(1) = a_{n,\varepsilon}^{d-p} \left[ \pi(\boldsymbol{\theta}_0) \int_{t(B_\delta) \times \mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{t} d\mathbf{v} + O_p(a_{n,\varepsilon}^{-1}) + O_p(a_n^2 \varepsilon_n^4) + O_p(\alpha_n^{-1}) \right],$$

$$\int_{t(B_\delta) \times \mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{t} d\mathbf{v} = \Theta_p(1),$$

(9)

$$\text{and } \frac{\pi_{B_\delta}(h)}{\pi_{B_\delta}(1)} = h(\boldsymbol{\theta}_0) + a_{n,\varepsilon}^{-1} Dh(\boldsymbol{\theta}_0)^T \left[ \frac{\int_{t(B_\delta) \times \mathbb{R}^d} \mathbf{t} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{t} d\mathbf{v}}{\int_{t(B_\delta) \times \mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{t} d\mathbf{v}} + O_p(a_{n,\varepsilon}^{-1}) + O_p(a_n^2 \varepsilon_n^4) + O_p(\alpha_n^{-1}) \right],$$

where  $t(B_\delta)$  is the transformed  $B_\delta$  under  $\mathbf{t}(\boldsymbol{\theta}) = a_{n,\varepsilon}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)$ .

PROOF. First consider  $\pi_{B_\delta}(1)$ . By (7), it only needs to evaluate  $\tilde{\pi}_{B_\delta}(1)$ . With the transformation  $\mathbf{t} = \mathbf{t}(\boldsymbol{\theta})$ ,

$$(10) \quad \tilde{\pi}_{B_\delta}(1) = a_{n,\varepsilon}^{-p} \int_{t(B_\delta) \times \mathbb{R}^d} \pi(\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t}) \tilde{f}_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v} | \boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t}) K(\mathbf{v}) d\mathbf{v} d\mathbf{t}$$

Since  $\tilde{f}_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v} | \boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t})$  is analytically available, we can obtain an expansion of  $\tilde{\pi}_{B_\delta}(1)$  by expanding  $\tilde{f}_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v} | \boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t})$  as follows. The expansion needs to be discussed in two cases as the limit of  $a_n \varepsilon_n$  being finite or infinite.

When  $a_n \varepsilon_n \rightarrow c_\varepsilon < \infty$ ,  $a_{n,\varepsilon} = a_n$ . Applying the Taylor expansion twice on the exponential term of  $\tilde{f}_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v} | \boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t})$ , both on  $a_n^{-1}$ , gives that

$$\begin{aligned} & \tilde{f}_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v} | \boldsymbol{\theta}_0 + a_n^{-1} \mathbf{t}) K(\mathbf{v}) \\ &= \frac{a_n^d |A(\boldsymbol{\theta}_0)|^{1/2}}{|A(\boldsymbol{\theta}_0 + a_n^{-1} \mathbf{t})|^{1/2}} \left[ g_n(\mathbf{t}, \mathbf{v}) + a_n^{-1} P_3(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}; r_1(\frac{\mathbf{t}}{a_n}), r_2(\frac{\mathbf{t}}{a_n})) \right], \end{aligned}$$

where  $r_1(\frac{\mathbf{t}}{a_n}) \mathbf{t} = \frac{1}{2} r_1 a_n^{-1} D_{t,2}(\mathbf{t}, \boldsymbol{\epsilon}_1(\mathbf{t}))$  for some  $|r_1| \leq 1$ ,  $r_2(\frac{\mathbf{t}}{a_n}) = r_2 a_n^{-1} D_{A,1}(\mathbf{t}, \boldsymbol{\epsilon}_2(\mathbf{t}))$  for some  $|r_2| \leq 1$ ,  $D_{t,2}(\mathbf{t}, \boldsymbol{\epsilon}_1(\mathbf{t}))$  is the  $d$ -dimension vector with the  $i_{th}$  element  $\mathbf{t}^T H \mathbf{s}_i(\boldsymbol{\theta}_0 + \boldsymbol{\epsilon}_1(\mathbf{t})) \mathbf{t}$

for some  $|\epsilon_1(\mathbf{t})| \leq \delta$ ,  $D_{A,1}(\mathbf{t}, \epsilon_2(\mathbf{t})) = \sum_{k=1}^p \frac{d}{d\theta_k} A(\boldsymbol{\theta}_0 + \epsilon_2(\mathbf{t}))^{-1/2} t_k$  for some  $|\epsilon_2(\mathbf{t})| \leq \delta$ , and the coefficients of  $P_3(\mathbf{t}, \mathbf{v})$  are  $O_p(1)$ . It can be seen that  $r_1(\mathbf{t}/a_n)$  and  $r_2(\mathbf{t}/a_n)$  are linear functions of  $\mathbf{t}/a_n$  with coefficients being  $O_p(1)$  and no constant terms, therefore can be arbitrarily small with small enough  $\delta$  since  $|\mathbf{t}|/a_n \leq \delta$ .

When  $a_n \varepsilon_n \rightarrow \infty$ ,  $a_{n,\varepsilon} = \varepsilon_n^{-1}$ . Let  $\mathbf{v}^* = A(\boldsymbol{\theta}_0)^{1/2} T_{obs} + a_n \varepsilon_n \mathbf{v} - a_n \varepsilon_n DS(\boldsymbol{\theta}_0) \mathbf{t}$ ,  $g_n^*(\mathbf{t}, \mathbf{v}^*; r_3(\varepsilon_n \mathbf{t}), r_4(\varepsilon_n \mathbf{t}))$  be  $g_n(\mathbf{t}, \mathbf{v}; r_3(\varepsilon_n \mathbf{t}), r_4(\varepsilon_n \mathbf{t}))$  transformed by  $\mathbf{v}^*$  and  $g_n^*(\mathbf{t}, \mathbf{v}^*) = g_n^*(\mathbf{t}, \mathbf{v}^*; 0, 0)$ . Applying the Taylor expansion twice on the exponential term, firstly on  $\varepsilon_n$  and secondly on  $a_n \varepsilon_n^2$ , gives that

$$\begin{aligned} & \tilde{f}_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v} | \boldsymbol{\theta}_0 + \varepsilon_n \mathbf{t}) K(\mathbf{v}) \\ &= \frac{\varepsilon_n^{-d} |A(\boldsymbol{\theta}_0)|^{1/2}}{|A(\boldsymbol{\theta}_0 + \varepsilon_n \mathbf{t})|^{1/2}} \left\{ g_n^*(\mathbf{t}, \mathbf{v}^*) + a_n \varepsilon_n^2 \left[ P_2(\mathbf{t}) \mathbf{v}^* + \frac{1}{a_n \varepsilon_n} P_1(\mathbf{t}) \mathbf{v}^* \mathbf{v}^{*T} \right] g_n^*(\mathbf{t}, \mathbf{v}^*) \right. \\ & \quad \left. + (a_n \varepsilon_n^2)^2 P_4(\mathbf{t}, \mathbf{v}^*) g_n^*(\mathbf{t}, \mathbf{v}^*; r_3(\varepsilon_n \mathbf{t}), r_4(\varepsilon_n \mathbf{t})) \right\} \end{aligned}$$

where  $r_3(\varepsilon_n \mathbf{t}) = \frac{r_3}{2} \varepsilon_n D_{t,2}(\mathbf{t}, \epsilon_1(\mathbf{t}))$  for some  $|r_3| \leq 1$  and  $r_4(\varepsilon_n \mathbf{t}) = r_4 \varepsilon_n D_{A,1}(\mathbf{t}, \epsilon_2(\mathbf{t}))$  for some  $|r_4| \leq 1$ . Similarly,  $r_3(\varepsilon_n \mathbf{t})$  and  $r_4(\varepsilon_n \mathbf{t})$  can be arbitrarily small with small enough  $\delta$  since  $\varepsilon_n \mathbf{t} \leq \delta$ .

Then plugging the above expansions and the following Taylor expansion

$$\frac{\pi(\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t})}{|A(\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t})|^{1/2}} = \frac{\pi(\boldsymbol{\theta}_0)}{|A(\boldsymbol{\theta}_0)|^{1/2}} + a_{n,\varepsilon}^{-1} D \frac{\pi(\boldsymbol{\theta}_0 + \epsilon_3(\mathbf{t}))}{|A(\boldsymbol{\theta}_0 + \epsilon_3(\mathbf{t}))|^{1/2}} \mathbf{t}, \text{ where } |\epsilon_3(\mathbf{t})| \leq \delta,$$

into the expression (10) of  $\tilde{\pi}_{B_\delta}(1)$ , it can be expanded as

$$\begin{aligned} & a_{n,\varepsilon}^{p-d} \tilde{\pi}_{B_\delta}(1) - \pi(\boldsymbol{\theta}_0) \int_{t(B_\delta) \times \mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{t} d\mathbf{v} \\ &= a_{n,\varepsilon}^{-1} \int_{t(B_\delta) \times \mathbb{R}^d} |A(\boldsymbol{\theta}_0)|^{1/2} D \frac{\pi(\boldsymbol{\theta}_0 + \epsilon_3(\mathbf{t}))}{|A(\boldsymbol{\theta}_0 + \epsilon_3(\mathbf{t}))|^{1/2}} \mathbf{t} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{t} d\mathbf{v} \\ & \quad + a_n^{-1} \int_{t(B_\delta) \times \mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}; r_1(a_{n,\varepsilon}^{-1} \mathbf{t}), r_2(a_{n,\varepsilon}^{-1} \mathbf{t})) d\mathbf{t} d\mathbf{v} \mathbb{1}_{\{a_{n,\varepsilon} = a_n\}} \\ & \quad + \varepsilon_n \int_{t(B_\delta)} P_3(\mathbf{t}) \int_{\mathbb{R}^d} (a_n \varepsilon_n)^2 \mathbf{v}^* g_n^*(\mathbf{t}, \mathbf{v}^*) d\mathbf{v}^* d\mathbf{t} \mathbb{1}_{\{a_{n,\varepsilon} = \varepsilon_n^{-1}\}} \\ & \quad + \varepsilon_n \int_{t(B_\delta) \times \mathbb{R}^d} P_2(\mathbf{t}) \mathbf{v}^* \mathbf{v}^{*T} g_n^*(\mathbf{t}, \mathbf{v}^*) d\mathbf{t} d\mathbf{v}^* \mathbb{1}_{\{a_{n,\varepsilon} = \varepsilon_n^{-1}\}} \\ & \quad + a_n^2 \varepsilon_n^4 \int_{t(B_\delta) \times \mathbb{R}^d} P_5(\mathbf{t}, \mathbf{v}^*) g_n^*(\mathbf{t}, \mathbf{v}^*; r_3(a_{n,\varepsilon}^{-1} \mathbf{t}), r_4(a_{n,\varepsilon}^{-1} \mathbf{t})) d\mathbf{t} d\mathbf{v}^* \mathbb{1}_{\{a_{n,\varepsilon} = \varepsilon_n^{-1}\}}. \end{aligned} \tag{11}$$

In the five terms in the RHS of above, the first two terms are  $O_p(a_{n,\varepsilon}^{-1})$  and  $O_p(a_n^{-1})$  and the fifth term is  $O_p(a_n^2 \varepsilon_n^4)$  by Lemma 4. The third term is  $O_p(\varepsilon_n)$  by noting that let  $\mathbf{e}_k = (0, \dots, 1, \dots, 0)$  with 1 at the  $k_{th}$  coordinate,

$$\int_{-\infty}^{\infty} v_k^* g_n^*(\mathbf{t}, \mathbf{v}^*) dv_k^* = \int_0^{\infty} v_k^* [g_n^*(\mathbf{t}, \mathbf{v}^*) - g_n^*(\mathbf{t}, \mathbf{v}^* - 2v_k^* \mathbf{e}_k)] dv_k^*$$

is bounded by  $(a_n \varepsilon_n)^{-2} c$  due to the symmetry of  $N(\mathbf{v}^*; 0, I_d)$ . The fourth term is obviously  $O_p(\varepsilon_n)$ . Then since  $\int_{t(B_\delta) \times \mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{t} d\mathbf{v} = \Theta_p(1)$  by Lemma 4, (8) holds.

Now consider  $\pi_{B_\delta}(h)/\pi_{B_\delta}(1)$ . By (5) and (7), we have

$$(12) \quad \frac{\pi_{B_\delta}(h)}{\pi_{B_\delta}(1)} = h(\boldsymbol{\theta}_0) + a_{n,\varepsilon}^{-1} Dh(\boldsymbol{\theta}_0)^T \left[ \frac{\tilde{\pi}_{B_\delta}(t(\boldsymbol{\theta}))}{\tilde{\pi}_{B_\delta}(1)} + O_p(\alpha_n^{-1}) \right] + \frac{1}{2} a_{n,\varepsilon}^{-2} \left[ \frac{\tilde{\pi}_{B_\delta}(t(\boldsymbol{\theta}))^T H h(\boldsymbol{\theta}_t) t(\boldsymbol{\theta}))}{\tilde{\pi}_{B_\delta}(1)} + O_p(\alpha_n^{-1}) \right]$$

Since  $Q_2(\mathbf{t})$  is a polynomial with bounded coefficients, where  $Q_2(\mathbf{t}) = \mathbf{t}$  or  $\mathbf{t} H h(\boldsymbol{\theta}_t) \mathbf{t}$ ,  $\tilde{\pi}_{B_\delta}(Q_2(\mathbf{t}))$  can be expanded similarly as for  $\tilde{\pi}_{B_\delta}(1)$ , simply multiplying  $g_n(\mathbf{t}, \mathbf{v})$  and  $P_4(\mathbf{t}, \mathbf{v})$  in (11) with  $Q_2(\mathbf{t})$ . Then it holds that  $\tilde{\pi}_{B_\delta}(Q_2(\mathbf{t})) = a_{n,\varepsilon}^{d-p} \left[ \pi(\boldsymbol{\theta}_0) \int_{t(B_\delta) \times \mathbb{R}^d} Q_2(\mathbf{t}) g_n(\mathbf{t}, \mathbf{v}) d\mathbf{t} d\mathbf{v} + O_p(a_{n,\varepsilon}^{-1}) + O_p(a_n^2 \varepsilon_n^4) \right]$ , and (9) holds by plugging this into the RHS of (12). Therefore the lemma holds.  $\square$

After obtaining the order of  $\tilde{\pi}_{B_\delta}(1)$ , (6) can be proved.

LEMMA 7. Assume conditions (C2) and (C4'). Then if  $\varepsilon_n = o(1/\sqrt{a_n})$ , (6) holds.

PROOF. Let  $r_{W_n}(\mathbf{w}|\boldsymbol{\theta}) = \alpha_n[f_{W_n}(\mathbf{w}|\boldsymbol{\theta}) - \tilde{f}_{W_n}(\mathbf{w}|\boldsymbol{\theta})]$ , and we have  $r_n(\mathbf{s}|\boldsymbol{\theta}) = a_n|A(\boldsymbol{\theta})|^{-1/2} r_{W_n}(a_n A(\boldsymbol{\theta})^{-1/2}(\mathbf{s} - \mathbf{s}(\boldsymbol{\theta}))|\boldsymbol{\theta})$ . Then since  $\pi(\boldsymbol{\theta})$  and  $A(\boldsymbol{\theta})$  are bounded for  $\boldsymbol{\theta} \in B_\delta$ , it is sufficient to show that

$$\int_{t(B_\delta)} \int P_l(\mathbf{t}, \mathbf{v}) (a_n a_{n,\varepsilon}^{-1})^d r_{max}(a_n a_{n,\varepsilon}^{-1} [D\mathbf{s}(\boldsymbol{\theta}_0 + e_t \mathbf{t}) \mathbf{t} - a_{n,\varepsilon} \varepsilon_n \mathbf{v} - a_n a_{n,\varepsilon}^{-1} A(\boldsymbol{\theta}_0)^{1/2} \mathbf{T}_{obs}]) K(\mathbf{v}) d\mathbf{v} d\mathbf{t} = O_p(1),$$

where the scalar  $e_t$  satisfying  $|e_t| \leq a_{n,\varepsilon}^{-1}$  and is from the Taylor expansion  $\mathbf{s}(\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t}) = \mathbf{s}(\boldsymbol{\theta}_0) + a_{n,\varepsilon}^{-1} D\mathbf{s}(\boldsymbol{\theta}_0 + e_t \mathbf{t}) \mathbf{t}$ . Note that when  $c_\varepsilon < \infty$ ,  $a_n a_{n,\varepsilon}^{-1} = 1$  and  $a_{n,\varepsilon} \varepsilon_n \rightarrow c_\varepsilon$ ; when  $c_\varepsilon = \infty$ ,  $a_n a_{n,\varepsilon}^{-1} \rightarrow 0$  and  $a_{n,\varepsilon} \varepsilon_n = 1$ . Therefore by Lemma 4, the lemma holds.  $\square$

PROOF. [Proof of Theorem 3.1] By applying Lemma 5 to the integral in (9), where the notations in Lemma 5 corresponds to

$$A = A(\boldsymbol{\theta}_0)^{-1/2} D\mathbf{s}(\boldsymbol{\theta}_0), \quad B_n = \begin{cases} a_n \varepsilon_n A(\boldsymbol{\theta}_0)^{-1/2}, & c_\varepsilon < \infty \\ A(\boldsymbol{\theta}_0)^{-1/2}, & c_\varepsilon = \infty \end{cases}, \quad k_n = \begin{cases} 1, & c_\varepsilon < \infty \\ a_n \varepsilon_n, & c_\varepsilon = \infty \end{cases} \text{ and } c = \mathbf{T}_{obs},$$

it can be seen that the integral has the order  $\Theta_p(a_{n,\varepsilon}/a_n)$ . Since  $\varepsilon_n = o(a_n^{-3/5})$  and  $\alpha_n^{-1} = o(a_n^{-2/5})$ , the other remainders are dominated. Then by Lemma 3 and Lemma 6, the leading term of  $a_n(\mathbf{h}_{ABC} - \mathbf{h}(\boldsymbol{\theta}_0))$  is

$$(13) \quad Dh(\boldsymbol{\theta}_0)^T \left( D\mathbf{s}(\boldsymbol{\theta}_0)^T A(\boldsymbol{\theta}_0)^{-1} D\mathbf{s}(\boldsymbol{\theta}_0) \right)^{-1} D\mathbf{s}(\boldsymbol{\theta}_0)^T A(\boldsymbol{\theta}_0)^{-1/2} \mathbf{T}_{obs} + r_n(\mathbf{T}_{obs}),$$

where  $r_n(\mathbf{T}_{obs}) = r(\mathbf{T}_{obs}; A(\boldsymbol{\theta}_0)^{-1/2} D\mathbf{s}(\boldsymbol{\theta}_0), a_{n,\varepsilon} \varepsilon_n A(\boldsymbol{\theta}_0)^{-1/2}, a_n a_{n,\varepsilon}^{-1})$  and  $r(\mathbf{c}; A, B, k_n)$  is defined in Lemma 5.  $r_n(\mathbf{T}_{obs})$  can be interpreted as the extra variation brought by  $\varepsilon_n$ , i.e.

$$(14) \quad r_n(\mathbf{T}_{obs}) = a_n(\mathbf{h}_{ABC} - E[\mathbf{h}(\boldsymbol{\theta})|\mathbf{s}_{obs}]),$$

since when  $\varepsilon_n = 0$ ,  $\mathbf{h}_{ABC} = \mathbb{E}[\mathbf{h}(\boldsymbol{\theta})|\mathbf{s}_{obs}]$  and  $r_n(\mathbf{T}_{obs}) = 0$ .

By delta method, (13) is asymptotically normal with mean  $r_n(0)$  and some covariance matrix, denoted by  $I_{ABC}(\boldsymbol{\theta}_0)$ . Since  $r_n(0) = 0$ , the asymptotic normality in (ii) holds. When  $d = p$ , since  $r_n(\mathbf{T}_{obs}) = 0$ ,  $I_{ABC}(\boldsymbol{\theta}_0) = I(\boldsymbol{\theta}_0)$ . When  $d > p$ , if  $\varepsilon_n = o(1/a_n)$ ,  $r_n(\mathbf{T}_{obs}) = o_p(1)$  and  $I_{ABC}(\boldsymbol{\theta}_0) = I(\boldsymbol{\theta}_0)$  holds; if  $a_n\varepsilon_n \rightarrow c_{\varepsilon_n} > 0$ ,  $r_n(\mathbf{T}_{obs})$  can not be ignored and  $I_{ABC}(\boldsymbol{\theta}_0)$  is not necessarily equal to  $I(\boldsymbol{\theta}_0)$ . Since  $I^{-1}(\boldsymbol{\theta}_0)$  is the Cramer-Rao lower bound,  $I_{ABC}(\boldsymbol{\theta}_0) \leq I(\boldsymbol{\theta}_0)$ .

For (i), the asymptotic normality holds for  $h(\hat{\boldsymbol{\theta}})$  by Lemma 2.  $\square$

## APPENDIX B: PROOF OF SECTION 4

The proof of Proposition 4.1 follows the standard asymptotic argument of importance sampling. For the detailed proof, see the supplement material.

For simplicity, consider one-dimension  $h(\boldsymbol{\theta})$ . Denote  $(h(\boldsymbol{\theta}) - h_{ABC})^2$  by  $G_n(\boldsymbol{\theta})$ . In Theorem 4.1(i),  $\Sigma_{IS,n}$  is just the ABC posterior variance of  $h(\boldsymbol{\theta})$ , and the derivation of its order is similar to that of  $h_{ABC}$  in Appendix A. The result is stated in the following lemma.

LEMMA 8. *Assume the conditions of Lemma 3. Then  $\text{Var}_{\pi_{ABC}}[h(\boldsymbol{\theta})] = O_p(a_{n,\varepsilon}^{-2})$ .*

PROOF. Using the notations of Appendix A,  $\text{Var}_{\pi_{ABC}}[h(\boldsymbol{\theta})] = \pi(G_n)/\pi(1)$ . It follows immediately from the arguments of Lemma 3 that  $\text{Var}_{\pi_{ABC}}[h(\boldsymbol{\theta})] = \pi_{B_\delta}(G_n)/\pi_{B_\delta}(1)(1 + o_p(1))$ . For its leading term, under the transformation  $\mathbf{t} = \mathbf{t}(\boldsymbol{\theta})$ , Taylor expansion of  $h(\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1}\mathbf{t})$  on  $a_{n,\varepsilon}^{-1}$  gives that

$$(15) \quad \frac{\pi_{B_\delta}(G_n)}{\pi_{B_\delta}(1)} = G_n(\boldsymbol{\theta}_0) + 2a_{n,\varepsilon}^{-1}(h(\boldsymbol{\theta}_0) - h_{ABC}) \frac{\pi_{B_\delta}(Dh(\boldsymbol{\theta}_t)^T \mathbf{t})}{\pi_{B_\delta}(1)} + a_{n,\varepsilon}^{-2} \frac{\pi_{B_\delta}(\mathbf{t}^T Dh(\boldsymbol{\theta}_t) Dh(\boldsymbol{\theta}_t)^T \mathbf{t}(\boldsymbol{\theta}))}{\pi_{B_\delta}(1)},$$

where  $\boldsymbol{\theta}_t \in B_\delta$ . In the above decomposition,  $G_n(\boldsymbol{\theta}_0)$  and  $a_{n,\varepsilon}^{-1}(h(\boldsymbol{\theta}_0) - h_{ABC})$  are  $O_p(a_{n,\varepsilon}^{-2})$  by Theorem 3.1. Then the lemma holds by the similar argument for (5).  $\square$

PROOF. [Proof of Theorem 4.1] For (i), since  $p_{acc,\pi} = \varepsilon_n^d \pi(1)$ , by Lemmas 3, 4 and 6,  $p_{acc,\pi} = \Theta_p(\varepsilon_n^d a_{n,\varepsilon}^{d-p})$  holds. Together by Lemma 8, (i) holds.

For (ii), if  $p_{acc,q} = \Theta_p(\varepsilon_n^d a_{n,\varepsilon}^d)$  holds, then by an alternative expression of  $\Sigma_{ABC,n}$

$$(16) \quad \Sigma_{ABC,n} = p_{acc,\pi}^{-1} E_{\pi_{ABC}} \left[ (\mathbf{h}(\boldsymbol{\theta}) - \mathbf{h}_{ABC})^2 \frac{\pi(\boldsymbol{\theta})}{q_n(\boldsymbol{\theta})} \right],$$

which can be verified easily by algebra, the order of  $\Sigma_{IS,n}$  is obvious. Similar to the expansion of  $\pi(1)$ ,

$$\begin{aligned}
p_{acc,q} &= \varepsilon_n^d \int \pi_{ABC}(\boldsymbol{\theta} | \mathbf{s}_{obs}, \varepsilon_n) f_{ABC}(\mathbf{s}_{obs} | \boldsymbol{\theta}) d\boldsymbol{\theta} \\
&= \varepsilon_n^d \frac{a_{n,\varepsilon}^{-p} \int_{t(B_\delta)} \pi(\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t}) \tilde{f}_{ABC}(\mathbf{s}_{obs} | \boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t})^2 d\mathbf{t}}{\tilde{\pi}_{B_\delta}(1)} (1 + o_p(1)).
\end{aligned}$$

The numerator of the above differs from  $\tilde{\pi}_{B_\delta}(1)$  by the square power of  $\tilde{f}_{ABC}(\mathbf{s}_{obs} | \boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t})$ . If the numerator has the order  $\Theta_p(a_{n,\varepsilon}^{2d})$ , then  $p_{acc,q} = \Theta_p(\varepsilon_n^d a_{n,\varepsilon}^d)$  would hold. By plugging in the expansions of  $\tilde{f}_n(\mathbf{s}_{obs} + \varepsilon_n \mathbf{v} | \boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1} \mathbf{t})$  in the proof of Lemma 6, an expansion of the numerator similar to (11) can be obtained, and the leading term would be of the order  $\Theta_p(a_{n,\varepsilon}^{2d})$  if the followings hold,

$$\begin{aligned}
&\int_{t(B_\delta)} \left( \int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v} \right)^2 d\mathbf{t} \text{ is } \Theta_p(1), \quad \int_{t(B_\delta)} \mathbf{t} \left( \int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v} \right)^2 d\mathbf{t} \\
(17) \quad &\text{and } \int_{t(B_\delta)} \left( \int_{\mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}; r_{1,3}(a_{n,\varepsilon}^{-1} \mathbf{t}), r_{2,4}(a_{n,\varepsilon}^{-1} \mathbf{t})) d\mathbf{v} \right)^2 d\mathbf{t} \text{ are } O_p(1).
\end{aligned}$$

To show that all the above integrals are  $O_p(1)$ , we only need that  $\int_{t(B_\delta)} \left( \int_{\mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v} \right)^2 d\mathbf{t} = O_p(1)$ . For the third integral in the above, its proof is similar by using the technique in Lemma ???. For the simplicity of presentation, assume  $A(\boldsymbol{\theta}_0) = I_d$  without loss of generality. The following arguments give the upper bound of  $\int_{\mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v}$ . Consider the two cases of  $g_n(\mathbf{t}, \mathbf{v})$ . When  $a_n \varepsilon_n \rightarrow c < \infty$ , let  $E_1 = \{\mathbf{v} : \|a_n \varepsilon_n \mathbf{v}\|^2 \leq \|D\mathbf{s}(\boldsymbol{\theta}_0)\mathbf{t} - \mathbf{T}_{obs}\|^2/2\}$ . Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v} \\
&= \left( \int_{E_1} + \int_{E_1^c} \right) P_4(\mathbf{t}, \mathbf{v}) \frac{1}{(2\pi)^{d/2}} \exp\left\{-\frac{1}{2}\|D\mathbf{s}(\boldsymbol{\theta}_0)\mathbf{t} - \mathbf{T}_{obs} - a_n \varepsilon_n \mathbf{v}\|^2\right\} K(\mathbf{v}) d\mathbf{v} \\
&\leq P_4(\mathbf{t}) \left[ \frac{1}{(2\pi)^{d/2}} \exp\left\{-\frac{1}{4}\|D\mathbf{s}(\boldsymbol{\theta}_0)\mathbf{t} - \mathbf{T}_{obs}\|^2\right\} + \frac{1}{(a_n \varepsilon_n)^d} K\left(\frac{1}{2a_n^2 \varepsilon_n^2} \|D\mathbf{s}(\boldsymbol{\theta}_0)\mathbf{t} - \mathbf{T}_{obs}\|^2\right) \right].
\end{aligned}$$

When  $a_n \varepsilon_n \rightarrow \infty$ , let  $E_2 = \{\mathbf{v} : \|\mathbf{v}\|^2 \leq \|D\mathbf{s}(\boldsymbol{\theta}_0)\mathbf{t} - (a_n \varepsilon_n)^{-1} \mathbf{T}_{obs}\|^2/2\}$ . Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v} \\
&= \left( \int_{E_2} + \int_{E_2^c} \right) P_4(\mathbf{t}, \mathbf{v}) \frac{(a_n \varepsilon_n)^d}{(2\pi)^{d/2}} \exp\left\{-\frac{a_n^2 \varepsilon_n^2}{2} \left\|D\mathbf{s}(\boldsymbol{\theta}_0)\mathbf{t} - \frac{1}{a_n \varepsilon_n} \mathbf{T}_{obs} - \mathbf{v}\right\|^2\right\} K(\mathbf{v}) d\mathbf{v} \\
&\leq P_4(\mathbf{t}) \left[ \frac{(a_n \varepsilon_n)^d}{(2\pi)^{d/2}} \exp\left\{-\frac{a_n^2 \varepsilon_n^2}{4} \|D\mathbf{s}(\boldsymbol{\theta}_0)\mathbf{t} - \frac{1}{a_n \varepsilon_n} \mathbf{T}_{obs}\|^2\right\} + K\left(\frac{1}{2} \left\|D\mathbf{s}(\boldsymbol{\theta}_0)\mathbf{t} - \frac{1}{a_n \varepsilon_n} \mathbf{T}_{obs}\right\|^2\right) \right].
\end{aligned}$$

In both cases, by the inequality  $(a+b)^2 \leq 2(a^2+b^2)$  and plugging in the upper bound, it is easy to see that  $\int_{t(B_\delta)} \left( \int_{\mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v} \right)^2 d\mathbf{t}$  is  $O_p(1)$ .

To see that the limit of  $\int_{t(B_\delta)} \left( \int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v} \right)^2 d\mathbf{t}$  is lower bounded away from 0, just using the positivity of the limit of the integrand and Fatou's lemma.  $\square$

Now let  $w_n(\boldsymbol{\theta})$  be the importance weight  $\pi(\boldsymbol{\theta})/q_n(\boldsymbol{\theta})$ ,  $\pi_{B_\delta, IS}(h) \triangleq \int_{B_\delta} h(\boldsymbol{\theta}) \pi(\boldsymbol{\theta}) f_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}) w_n(\boldsymbol{\theta}) d\boldsymbol{\theta}$  and  $\pi_{B_\delta^c, IS}(h)$  correspondingly. Then by the expression (16),

$$(18) \quad \Sigma_{ABC, n} = p_{acc, \pi}^{-1} \frac{\pi_{B_\delta, IS}(G_n) + \pi_{B_\delta^c, IS}(G_n)}{\pi_{B_\delta}(1) + \pi_{B_\delta^c}(1)}.$$

PROOF. [Proof of Theorem 4.2] For  $p_{acc, q_n}$ , we only need to consider  $\beta = 1$ . Using the transformation  $\mathbf{t} = \mathbf{t}(\boldsymbol{\theta}) \triangleq a_{n, \varepsilon}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)$ , since  $a_{n, \varepsilon} \sigma_n = 1$ ,  $q_n(\boldsymbol{\theta}) = a_{n, \varepsilon}^p |\Sigma|^{1/2} q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu))$ . Then similar to the approximation of  $\pi(1)$ ,

$$\begin{aligned} p_{acc, q_n} &= \varepsilon_n^d \int q_n(\boldsymbol{\theta}) f_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}) d\boldsymbol{\theta} \\ &= \varepsilon_n^d a_{n, \varepsilon}^p |\Sigma|^{1/2} \int_{t(B_\delta)} q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu)) \tilde{f}_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}_0 + a_{n, \varepsilon}^{-1} \mathbf{t}) d\mathbf{t} (1 + o_p(1)). \end{aligned}$$

Plugging in the expansions of  $\tilde{f}_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}_0 + a_{n, \varepsilon}^{-1} \mathbf{t})$  in Lemma 6, we can obtain the expansion similar to (11), and it can be seen that  $p_{acc, q_n} = \Theta_p(a_{n, \varepsilon}^d \varepsilon_n^d)$  if  $\int_{\mathbb{R}^d \times t(B_\delta)} q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu)) g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v} d\mathbf{t} = \Theta_p(1)$  and  $\int_{\mathbb{R}^d \times t(B_\delta)} q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu)) g_n(\mathbf{t}, \mathbf{v}; \tau_1, \tau_2) d\mathbf{v} d\mathbf{t} = O_p(1)$ , where  $\tau_1^T \tau_1 \leq \delta I_p$  and  $\tau_2^T \tau_2 \leq \delta I_d$ . Noting that  $q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu))$  is upper bounded for  $t \in \mathbb{R}^p$ , by Lemma ?? and Lemma 4, these two integrals are upper bounded. By the positivity of the limit of the integrand and Fatou's lemma, the first integral is lower bounded. Therefore  $p_{acc, q_n} = \Theta_p(a_{n, \varepsilon}^d \varepsilon_n^d)$  holds.

For  $\Sigma_{IS, n}$ , by its definition and (18), we have

$$\Sigma_{IS, n} = p_{acc, q_n} \Sigma_{ABC, n} = \frac{p_{acc, q_n}}{p_{acc, \pi}} \frac{\pi_{B_\delta, IS}(G_n)}{\pi_{B_\delta}(1)} (1 + o_p(1)),$$

where the second equality holds since  $\pi_{B_\delta^c, IS}(G_n)$  is ignorable by noting that  $\omega_n(\boldsymbol{\theta}) \leq \beta^{-1}$  and using the arguments of Lemma 3. Given the orders of  $p_{acc, q_n}$  and  $p_{acc, \pi}$  which are already obtained, in order for  $\Sigma_{IS, n} = O_p(a_{n, \varepsilon}^{-2})$ , it only needs  $\pi_{B_\delta, IS}(G_n)/\pi_{B_\delta}(1) = O_p(a_{n, \varepsilon}^{p-2})$ . Similar to (15), we have the following expansion

$$\frac{\pi_{B_\delta, IS}(G_n)}{\pi_{B_\delta}(1)} = G(\boldsymbol{\theta}_0) \frac{\pi_{B_\delta, IS}(1)}{\pi_{B_\delta}(1)} + 2a_{n, \varepsilon}^{-1} (h(\boldsymbol{\theta}_0) - h_{ABC}) \frac{\pi_{B_\delta, IS}(Dh(\boldsymbol{\theta}_t)^T \mathbf{t})}{\pi_{B_\delta}(1)} + a_{n, \varepsilon}^{-2} \frac{\pi_{B_\delta, IS}(\mathbf{t}^T Dh(\boldsymbol{\theta}_t) Dh(\boldsymbol{\theta}_t)^T \mathbf{t})}{\pi_{B_\delta}(1)},$$

and we only need  $\pi_{B_\delta, IS}(P_2(\mathbf{t}))/\pi_{B_\delta}(1) = O_p(a_{n, \varepsilon}^p)$ . Since  $w_n(\boldsymbol{\theta}) \leq (1 - \beta)^{-1} w_{n, 1}(\boldsymbol{\theta})$ , where  $w_{n, 1}(\boldsymbol{\theta})$  is the weight when  $\beta = 1$ , it is sufficient to consider the case  $\beta = 1$ . By (7),  $\pi_{B_\delta, IS}$



can be replaced by  $\tilde{\pi}_{B_\delta, IS}$  in which  $f_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta})$  is replaced by  $\tilde{f}_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta})$ . Using the transformation  $\mathbf{t}(\boldsymbol{\theta})$  and plugging in the expansions of  $\tilde{f}_{ABC}(\mathbf{s}_{obs}|\boldsymbol{\theta}_0 + a_{n,\varepsilon}^{-1}\mathbf{t})$ , we have the following expansion similar to (11),

$$\begin{aligned} & a_{n,\varepsilon}^{-d} \pi_{B_\delta, IS}(P_2(\mathbf{t})) - \pi(\boldsymbol{\theta}_0)^2 \int_{t(B_\delta)} P_2(\mathbf{t}) \frac{\int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v}}{q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu))} d\mathbf{t} \\ &= a_{n,\varepsilon}^{-1} \int_{t(B_\delta)} |A(\boldsymbol{\theta}_0)|^{1/2} D \frac{\pi(\boldsymbol{\theta}_0 + \epsilon_3(\mathbf{t}))^2}{|A(\boldsymbol{\theta}_0 + \epsilon_3(\mathbf{t}))|^{1/2}} P_3(\mathbf{t}) \frac{\int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v}}{q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu))} d\mathbf{t} \\ &+ \left( a_n \mathbb{1}_{\{a_{n,\varepsilon}=a_n\}} + a_n \varepsilon_n^2 \mathbb{1}_{\{a_{n,\varepsilon}=\varepsilon_n^{-1}\}} \right) \int_{t(B_\delta)} P_2(\mathbf{t}) \frac{\int_{\mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}; r_{1,3}(a_{n,\varepsilon}^{-1}\mathbf{t}), r_{2,4}(a_{n,\varepsilon}^{-1}\mathbf{t})) d\mathbf{v}}{q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu))} d\mathbf{t}. \end{aligned}$$

Then we need to show that

$$(19) \quad \int_{t(B_\delta)} P_3(\mathbf{t}) \frac{\int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v}}{q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu))} d\mathbf{t} \text{ and } \int_{t(B_\delta)} P_2(\mathbf{t}) \frac{\int_{\mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}; r_{1,3}(a_{n,\varepsilon}^{-1}\mathbf{t}), r_{2,4}(a_{n,\varepsilon}^{-1}\mathbf{t})) d\mathbf{v}}{q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu))} d\mathbf{t} \text{ are } O_p(1).$$

By (C10), for the first integral in the above, we have

$$\frac{\int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v}}{q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu))} \leq M_n \left( \int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}) d\mathbf{v} \right)^{1-\alpha},$$

where  $M_n$  is a scalar and has the order  $O_p(1)$ ; for the second integral,

$$\begin{aligned} & \frac{\int_{\mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) g_n(\mathbf{t}, \mathbf{v}; r_{1,3}(a_{n,\varepsilon}^{-1}\mathbf{t}), r_{2,4}(a_{n,\varepsilon}^{-1}\mathbf{t})) d\mathbf{v}}{q(\Sigma^{-1/2}(\mathbf{t} - \mathbf{c}_\mu))} \\ & \leq M_n \int_{\mathbb{R}^d} P_4(\mathbf{t}, \mathbf{v}) \frac{g_n(\mathbf{t}, \mathbf{v}; r_{1,3}(a_{n,\varepsilon}^{-1}\mathbf{t}), r_{2,4}(a_{n,\varepsilon}^{-1}\mathbf{t}))}{\int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}; r_{1,3}(a_{n,\varepsilon}^{-1}\mathbf{t}), r_{2,4}(a_{n,\varepsilon}^{-1}\mathbf{t})) d\mathbf{v}} d\mathbf{v} \left( \int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}; r_{1,3}(a_{n,\varepsilon}^{-1}\mathbf{t}), r_{2,4}(a_{n,\varepsilon}^{-1}\mathbf{t})) d\mathbf{v} \right)^{1-\alpha} \\ & = P_4(\mathbf{t}) \left( \int_{\mathbb{R}^d} g_n(\mathbf{t}, \mathbf{v}; r_{1,3}(a_{n,\varepsilon}^{-1}\mathbf{t}), r_{2,4}(a_{n,\varepsilon}^{-1}\mathbf{t})) d\mathbf{v} \right)^{1-\alpha}. \end{aligned}$$

Then by the inequality  $(a+b)^{1-\alpha} \leq a^{1-\alpha} + b^{1-\alpha}$ , for  $\alpha \in (0, 1)$  and  $a, b > 0$ , and following the arguments of (17), (19) holds. Therefore  $\Sigma_{IS,n} = O_p(a_{n,\varepsilon}^{-2})$ .  $\square$

PROOF. [Proof of Proposition 4.2]

With the notations of Lemma 4, it is sufficient to show that

$$\sup_{\mathbf{t} \in \mathbb{R}^p, \tau^T \tau \leq \delta I_p} \frac{\left[ \int N\left((A + \tau)\mathbf{t}; B\mathbf{v} + \mathbf{c}, \frac{1}{k_n^2} I_d\right) K(\mathbf{v}) d\mathbf{v} \right]^\alpha}{T(\Sigma^{-1/2}(\mathbf{t} - \boldsymbol{\mu}); \gamma)} = O_p(1).$$

This can be seen by the inequality (7) in the supplement material, the inequality  $(a+b)^\alpha \leq a^\alpha + b^\alpha$  and the assumption on  $K(\mathbf{v})^\alpha$ .  $\square$

DEPARTMENT OF MATHEMATICS AND STATISTICS

E-MAIL: [w.li@lancaster.ac.uk](mailto:w.li@lancaster.ac.uk)

E-MAIL: [p.fearnhead@lancaster.ac.uk](mailto:p.fearnhead@lancaster.ac.uk)